The Moduli Space of Vacua of $N = 2$ SUSY QCD and Duality in $N = 1$ SUSY QCD

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We analyze in detail the moduli space of vacua of N=2 SUSY QCD with n_c colors and n_f flavors. The Coulomb branch has submanifolds with non-Abelian gauge symmetry. The massless quarks and gluons at these vacua are smoothly connected to the underlying elementary quarks and gluons. Upon breaking N=2 by an N=1 preserving mass term for the adjoint field the theory flows to N=1 SUSY QCD. Some of the massless quarks and gluons on the moduli space of the N=2 theory become the magnetic quarks and gluons of the N=1 theory. In this way we derive the duality in N=1 SUSY QCD by identifying its crucial building blocks—the magnetic degrees of freedom—using only semiclassical physics and the non-renormalization theorem.

1. Introduction and Summary

In the last few years it has become clear that supersymmetric field theories in four dimensions can be analyzed exactly (for a review and a list of references see [1,2]). The main new dynamical insight which has been gained is the role of electric-magnetic duality.

In its simplest form duality is an exact equivalence between theories [3]. This is the case in the free Abelian theory, in $N=4$ SUSY theories [4] and in finite $N=2$ theories [5]. Asymptotically free theories are unlikely to exhibit such exact dualities. However, if the low energy theory is an Abelian theory, its duality appears as an ambiguity in the low energy effective Lagrangian. This ambiguity in the description plays a crucial role in solving the theory [6]. An alternative notion of duality for asymptotically free theories was suggested in [7] where two different dual theories exhibit the same long distance behavior.

It is important to stress that (with the exception of the duality in the free Abelian theory) neither the exact Montonen-Olive duality [3] nor the duality in $N=1$ theories [7] have been proven. There is a lot of evidence that these dualities are true. Furthermore, under the renormalization group one can flow from a dual pair of $N=1$ theories to a dual pair of $N=4$ theories [8], implying that the duality in $N=1$ is a generalization of the duality in $N=4$.

One of the motivations of this paper is to extend this understanding, by showing that many of the crucial elements in the duality of [7] in $N=1$ SUSY theories can be traced back to the corresponding $N=2$ SUSY theories; for earlier work along these lines, see [9]. In particular, the elementary *electric quarks and gluons* of the $N=2$ theory can be continuously deformed to the *magnetic quarks and gluons* of the $N=1$ theory. This explicit demonstration of their existence is close to a proof of the duality of [7].

The fact that electric quarks and gluons can be continuously deformed to magnetic quarks and gluons sounds surprising at first. However, it is in accord with the general principles of [10], where it is shown that one can continuously interpolate between the Higgs and the confinement phases in theories with matter fields in the fundamental representation. Since these two phases are obtained by the condensation of electric and magnetic charges respectively, one should be able to continuously deform electric charges to magnetic charges in such theories. This was explicitly demonstrated in [5] for Abelian charges

and photons. Here we extend it to electric and magnetic non-Abelian quarks and gluons.

We start our analysis in section 2 by studying at the classical level N=2 SUSY QCD based on an $SU(n_c)$ gauge theory with n_f quark hypermultiplets in the fundamental representation. The moduli space of classical vacua consists of a Coulomb branch where the gauge group is of rank n_c-1 and various Higgs branches where the gauge group is of lower rank. The different branches touch each other at singular points where new massless particles are present. It will turn out to be crucial that for $n_c \leq n_f \leq 2n_c-2$ the theory has distinct Higgs branches touching each other at singular points as shown in Fig. 1.
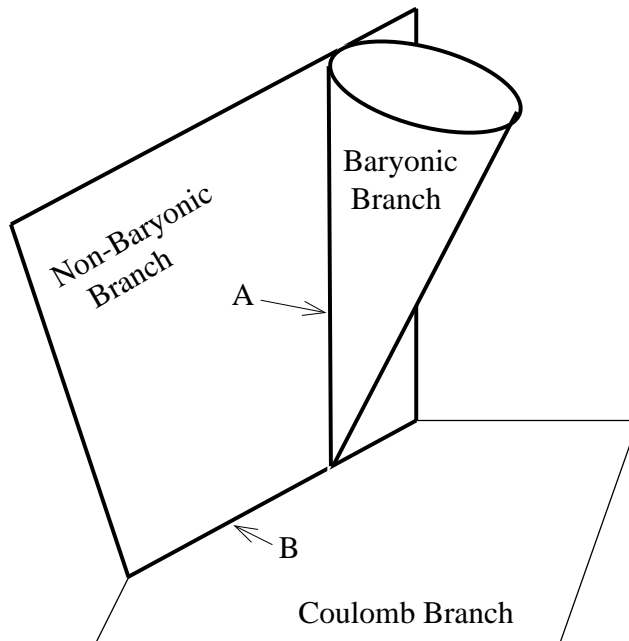

Fig. 1: Map of the classical moduli space of N=2 $SU(n_c)$ QCD with n_f fundamental flavors. The baryonic and non-baryonic Higgs branches intersect along a submanifold A, while the non-baryonic branch intersects the Coulomb branch along submanifold B where there is an unbroken $SU(r) \times U(1)^{n_c-r}$ gauge symmetry with n_f massless fundamental hypermultiplets. A and B intersect at a point where the full $SU(n_c)$ with n_f hypermultiplets is unbroken. There are separate non-baryonic branches for $1 \leq r \leq [n_f/2]$.

We divide the various Higgs branches into baryonic and non-baryonic branches, names following from the fact that on the non-baryonic branches all the light fields have vanishing baryon number. There will turn out to be a single baryonic branch for $n_f \geq n_c$ whose generic low-energy effective theory consists of $n_f n_c - n_c^2 + 1$ massless hypermultiplets. Non-baryonic branches will be shown to exist for $n_f \geq 2$, each with (generically) $n_c - 1 - r$

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massless vector multiplets corresponding to a $U(1)^{n_c-1-r}$ low energy gauge group, and $r(n_f-r)$ massless neutral hypermultiplets, where $1 \le r \le \min\{[n_f/2], n_c-2\}$. (Only one non-baryonic branch is shown in Fig. 1 due to lack of dimensions.)

In section 3 we give a very general argument showing that the Higgs branches as determined classically cannot be modified in the quantum theory. This means that the only possible quantum modification is that distinct Higgs branches which touch classically may separate, touching the quantum Coulomb branch at different points due to the splitting of points on the Coulomb branch. For example, classically the baryonic and non-baryonic branches meet at the origin of the Coulomb branch and also intersect along a submanifold. Since the origin is split quantum-mechanically, it is possible that the baryonic branch is split from the non-baryonic branch. Since, by the non-renormalization theorem, the Coulomb part of a branch cannot depend on the squark vevs (*i.e.*, on where it attaches to the Higgs branch), it follows that the baryonic and non-baryonic branches must be completely disjoint quantum-mechanically (see Fig. 2). Such a phenomenon has already been observed in [5] for $n_c = n_f = 2$ and here we demonstrate it for other values of n_c and n_f.

Next we analyze the Coulomb branch at weak coupling. Semiclassically the Coulomb branch is characterized by n_c complex numbers (up to permutation) whose sum vanishes. These are the eigenvalues $(\phi_1, ..., \phi_{n_c})$ of the complex scalar field Φ in the adjoint representation of the gauge group. At the generic point on the moduli space where all the eigenvalues are different, the $SU(n_c)$ gauge symmetry is broken to its Cartan subalgebra $U(1)^{n_c-1}$. The superpotential $\text{Tr}\widetilde{Q}\Phi Q$ which couples the adjoint field Φ to the quarks Q and $\widetilde{Q}$ makes all the quarks massive at these points. On submanifolds where k of the eigenvalues are equal $SU(n_c)$ is broken to $SU(k) \times U(1)^{n_c-k}$ and the low energy semiclassical theory includes more gluons. On submanifolds where some eigenvalues vanish the classical theory has also n_f massless quarks (the term in the superpotential $\text{Tr}\widetilde{Q}\Phi Q$ does not give them a mass). Therefore, when k eigenvalues vanish the low energy theory is $SU(k) \times U(1)^{n_c-k}$ with n_f massless quarks in the fundamental representation of $SU(k)$. If $2k < n_f$ the low energy theory is IR–free and the effective coupling constant of the massless quarks and gluons vanishes at long distance. Therefore, the massless quarks and
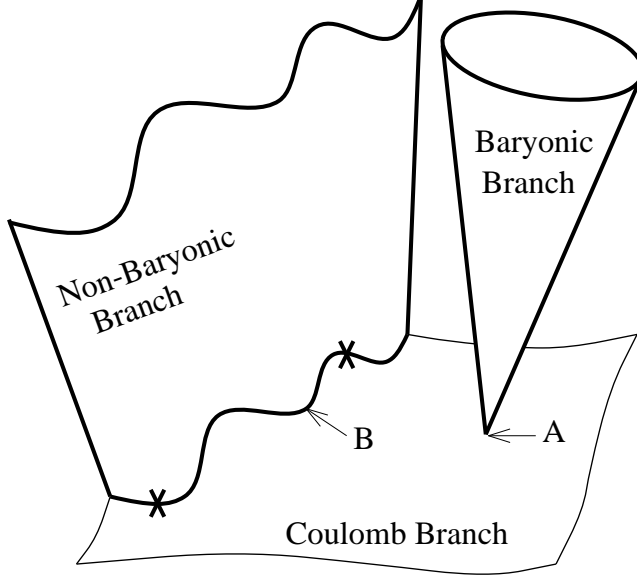
Fig. 2: Map of the quantum moduli space of $N{=}2$ $SU(n_c)$ QCD with n_f fundamental flavors. Point A has unbroken gauge group $SU(n_f{-}n_c)$ with n_f massless fundamental hypermultiplets as well as various extra monopole singlets. Submanifold B has unbroken gauge group $SU(r) \times U(1)^{n_c-r}$ with n_f fundamental hypermultiplets. The X's mark points (submanifolds) on B where there are extra massless singlets.

gluons remain in the spectrum in the full quantum theory. Similarly, for $2k = n_f$ the low energy theory is finite. The coupling constant does not grow at long distance and the massless quarks and gluons which were found in the classical analysis will remain massless (though interacting) in the quantum theory. This semiclassical analysis is valid for $\Phi = (0, \ldots, 0, \phi_{k+1}, \ldots, \phi_{n_c})$ with all $|\phi_i| \gg \Lambda$. But given this analysis, these submanifolds of vacua with massless gluons can be followed in to strong coupling.

We thus conclude that on certain submanifolds of the Coulomb branch the exact quantum theory must have IR–free massless quarks and gluons or massless interacting quarks and gluons of a finite $N{=}2$ theory[1]. Some of these will later turn out to be the magnetic quarks and gluons of the corresponding $N{=}1$ theory.

In section 4 we determine the spectra of massless particles at the vacua where the various Higgs branches meet the Coulomb branch (the "roots" of the Higgs branches). The physics along these submanifolds of the Coulomb branch includes the IR–free quarks

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[1] Non-trivial interacting fixed points such as those of [11,12] can also exist.

and gluons described above. In the case of the non-baryonic branches the generic theory at the root is the IR–free or finite $SU(r) \times U(1)^{n_c-r}$ QCD with n_f quark hypermultiplets in the fundamental representation and charged under one of the $U(1)$ factors with $2r \leq n_f$. There are special points on the submanifolds comprising these non-baryonic roots where n_c-r-1 additional singlet hypermultiplets charged under the $U(1)$ factors become massless (see Fig. 2). If we call the singlets e_i, then, by an appropriate change of basis of the $U(1)$'s, their charges can be taken to be

$$
\begin{array}{ccccccccccccc}
 & SU(r) & \times & U(1)_0 & \times & U(1)_1 & \times & \cdots & \times & U(1)_{n_c-r-1} & \times & U(1)_B \\
n_f \times q & \mathbf{r} & & 1 & & 0 & & \cdots & & 0 & & 0 \\
e_1 & \mathbf{1} & & 0 & & 1 & & \cdots & & 0 & & 0 \\
\vdots & \vdots & & \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
e_{n_c-r-1} & \mathbf{1} & & 0 & & 0 & & \cdots & & 1 & & 0
\end{array}
\tag{1.1}
$$

Just as we have picked a convenient basis for the gauge charges, we have also used the freedom to shift the global $U(1)_B$ baryon number by an arbitrary gauge charge. Our choice is such that the baryon charges of all the light fields vanish—hence the name of these branches. There is also a global $SU(n_f) \times SU(2)_R$ symmetry which is not included in (1.1).

Since the root of the baryonic branch is a single point, it must be invariant under the discrete global $\mathbb{Z}_{2n_c-n_f}$ symmetry of the theory (the anomaly-free part of the classical $U(1)$ R-symmetry). This suggests that the coordinates of the root on the Coulomb branch are $\Phi \propto (0,\ldots,0,\omega^1,\ldots,\omega^{2n_c-n_f})$ where ω is a $2n_c-n_f$-th root of unity. The $\widetilde{n}_c \equiv n_c-n_f$ zeros imply an unbroken (and IR–free) $SU(\widetilde{n}_c) \times U(1)^{2n_c-n_f}$ gauge group. The requirement that there be a Higgs branch emanating from this vacuum implies that there also be $2n_c-n_f$ massless singlet hypermultiplets charged under the $U(1)$ factors. By an appropriate change of basis for the $U(1)$'s, the charges can be taken to be

$$
\begin{array}{ccccccccccc}
 & SU(\widetilde{n}_c) & \times & U(1)_1 & \times & \cdots & \times & U(1)_{2n_c-n_f} & \times & U(1)_B \\
n_f \times q & \widetilde{\mathbf{n}}_{\mathbf{c}} & & 1/\widetilde{n}_c & & \cdots & & 1/\widetilde{n}_c & & -n_c/\widetilde{n}_c \\
e_1 & \mathbf{1} & & -1 & & \cdots & & 0 & & 0 \\
\vdots & \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
e_{2n_c-n_f} & \mathbf{1} & & 0 & & \cdots & & -1 & & 0
\end{array}
\tag{1.2}
$$

(Here again we have included the global $U(1)_B$ baryon number charges, while the $SU(n_f) \times SU(2)_R$ part of the global symmetry was not included.) We check that (1.2) is the correct

identification of the spectrum at the baryonic branch root by showing that the Higgs branch emanating from this special vacuum is identical to the baryonic Higgs branch determined in the classical theory. As a byproduct, this argument determines the baryon number of the quarks shown in (1.2). This spectrum can also be determined by starting with the finite $n_f = 2n_c$ theory and using its conjectured duality to flow down in flavors by giving masses to quarks.

In section 5 we review the detailed description of the Coulomb branch found for the pure gauge $SU(n_c)$ theory in [13,14] and for the theory with matter in [15,16].[2] In the quantum theory the moduli space is also characterized by n_c complex numbers ϕ_i (up to permutations) whose sum vanishes. The coupling constants of the low energy gauge fields form a section of an $Sp(n_c-1, \mathbb{Z})$ bundle over the moduli space [6] which was determined explicitly in [13-16]. This allows us to track the special (singular) submanifolds with enhanced non-Abelian symmetries found previously into the strong coupling regime. In particular, we find the explicit coordinates of the non-baryonic and baryonic roots on the Coulomb branch. The monodromies around these singular points enable us to check the spectrum of massless particles at the singularities (1.1) and (1.2).

In section 6 we use the answers of the previous sections to flow from $N{=}2$ to $N{=}1$ by giving a mass μ to the adjoint $N{=}1$ superfield Φ. Integrating Φ out of the microscopic theory we find for $\mu{\gg}\Lambda$, $N{=}1$ $SU(n_c)$ QCD with n_f flavors describing the theory at scales below μ but above the strong-coupling scale of the $N{=}1$ theory, given by a one-loop matching as

$$\Lambda_1^{3n_c-n_f} = \mu^{n_c}\Lambda^{2n_c-n_f} \ . \tag{1.3}$$

To find the extreme low-energy limit, however, we can also first integrate out the degrees of freedom with mass of order Λ. For $\mu{\ll}\Lambda$ this leads to the effective theory described in the previous sections. We can then study the breaking to $N{=}1$ as a deformation of this theory. We find that generic vacua in the moduli space are lifted by this perturbation, but we show that special points along the roots of the Higgs branches are not. Thus the massless fields in these vacua will descend to the $N{=}1$ theory. (This analysis cannot rule

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[2] The scale-invariant $SU(3)$ solution found in [17] is equivalent to that of [15], being related by a reparametrization of the bare coupling—see [18].

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out the presence of other fields which become light only in the $\mu\to\infty$ limit.) The gauge singlets in (1.2) and (1.1) and the squarks in (1.1) condense and thus reduce the gauge symmetry. However, it is crucial that the squarks in (1.2) do not condense and therefore the $SU(n_f-n_c)$ quarks and the gluons of equation (1.2) remain massless. They become the "magnetic" quarks and gluons of [7]. Furthermore, in the limit $\mu\to\infty$ the two branches merge and the light gauge singlet fields from the the non-baryonic branches contribute to the extra singlet fields found in the dual $N{=}1$ theory [7].

Thus, we have shown that the extreme low-energy limit of the $N{=}2$ theory is obtained by starting at intermediate scales with either of the two dual descriptions of [7] (see Fig. 3). This is close to a proof of the duality found in that work. We note that by explicitly identifying the magnetic degrees of freedom at an intermediate scale where they are weakly coupled we prove the exact low energy equivalence (not merely identifying the two chiral rings) of the two dual $N{=}1$ theories. Moreover, in the $N{=}2$ theory we find that the "magnetic" quarks and gluons are continuously connected to the "electric" degrees of freedom, by following the singularity associated to their becoming massless from the weak coupling region in to the strongly-coupled points where the extra massless gauge singlets appear.

It is interesting to note how the global symmetries and quantum numbers are related. In the $N{=}1$ theory the global symmetry is $SU(n_f)\times SU(n_f)\times U(1)_B\times U(1)_R$. For finite μ the $U(1)_R$ symmetry is broken and the $SU(n_f)\times SU(n_f)$ is broken to its diagonal $SU(n_f)$ subgroup. The extra symmetries of the $\mu\to\infty$ limit appear as "accidental" symmetries of the limiting theory. When $\mu\to0$ the resulting $N{=}2$ theory has global symmetry $SU(n_f)\times U(1)_B\times SU(2)_R$, where the last factor is present only for $\mu{=}0$. In relating the two $N{=}1$ theories the quarks of the electric theory can switch their $SU(n_f)\times SU(n_f)$ quantum numbers to become the magnetic dual quarks since for finite μ this symmetry is broken to the diagonal $SU(n_f)$ under which they transform in the same way. Their baryon numbers change due to mixing with various gauge $U(1)$ quantum numbers as we follow the $N{=}2$ vacua in to strong coupling. In particular, when $\mu\neq0$ the various singlets in (1.1) and (1.2) condense, breaking some of the global symmetries. We have defined the $U(1)_B$ in (1.1) and (1.2) to be the combination of the original baryon number with the $U(1)$ gauge

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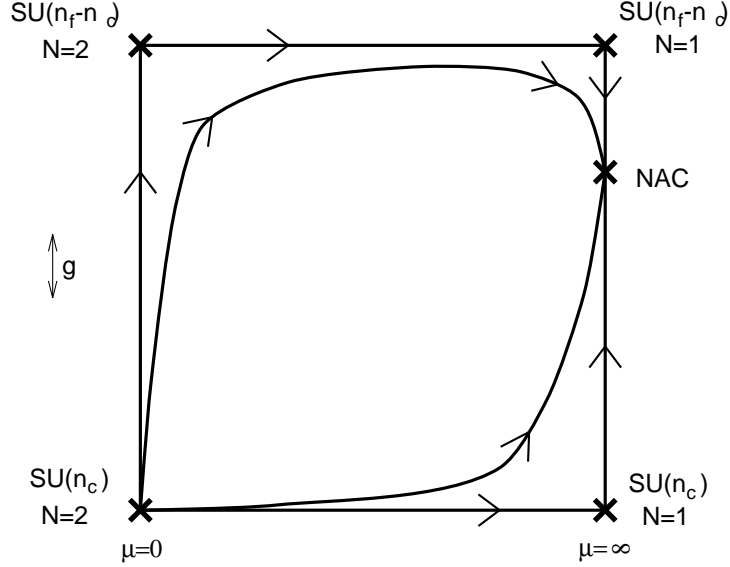
Fig. 3: Renormalization group flows in coupling constant and μ. The two trajectories depicted represent the two approaches to the IR fixed point which approximate the "electric" and "magnetic" $N{=}1$ theories at intermediate scales. The diagram holds for $n_f \geq 3/2 n_c$. For smaller n_f the magnetic $N{=}1$ theory is IR free and the non-Abelian Coulomb (NAC) fixed point coincides with the upper-right-hand corner. The four corners are labeled by the gauge group and supersymmetry; all have n_f families of quarks. The models on the top edge have additional massless singlets as discussed in the text.

generators which is left unbroken for non-zero μ.

2. Classical Moduli Space

2.1. Symmetries and Vacuum Equations

$N{=}2$ $SU(n_c)$ supersymmetric QCD is described in terms of $N{=}1$ superfields by a field strength chiral multiplet W_α and a scalar chiral multiplet Φ both in the adjoint of the gauge group, and chiral multiplets Q^i in the $\mathbf{n_c}$, and $\widetilde{Q}_i$ in the $\mathbf{\overline{n}_c}$ representations of the gauge group, where $i = 1, \ldots, n_f$ are flavor indices. W and Φ together form the $N{=}2$ vector multiplet, while Q and $\widetilde{Q}$ make up a hypermultiplet. We denote the complex scalar components of Φ, Q, and $\widetilde{Q}$ and their vevs by the same symbols. The Lagrangian in terms of $N{=}1$ superfields is (we follow the conventions of [19])

$$4\pi\mathcal{L} = \text{Im}\left[\tau \int d^2\theta\, d^2\overline{\theta}\, \text{tr}\left(\Phi^\dagger e^V \Phi + Q_i^\dagger e^V Q^i + \widetilde{Q}^{\dagger i} e^V \widetilde{Q}_i\right) + \tau \int d^2\theta \left(\tfrac{1}{2}\text{tr}W^2 + \mathcal{W}\right)\right], \quad (2.1)$$

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where τ is the gauge coupling constant and $\mathcal{W}$ is the $N{=}2$ superpotential

$$\mathcal{W} = \sqrt{2}\widetilde{Q}_i^a \Phi_a^b Q_b^i + \sqrt{2}m_j^i \widetilde{Q}_i^a Q_a^j, \tag{2.2}$$

$a, b = 1, \ldots, n_c$ are color indices, and m_j^i is a quark mass matrix which must satisfy

$$[m, m^\dagger] = 0 \tag{2.3}$$

to preserve $N{=}2$ supersymmetry. The condition (2.3) implies that m can be diagonalized by a flavor rotation to a complex diagonal matrix $m = \mathrm{diag}(m_1, \ldots, m_{n_f})$. Classically and with no masses the global symmetries are a $U(1)_B \times SU(n_f)$ flavor symmetry and a $U(1)_R \times SU(2)_R$ chiral R-symmetry which acts on the $N{=}2$ supersymmetry algebra. Mass terms and instanton corrections break $U(1)_R$. Under the unbroken $SU(2)_R$ the vector A_μ and the scalar Φ are singlets, while the vector multiplet fermions form a doublet. Similarly for the hypermultiplets, the fermions are singlets, while their scalar partners Q and $\widetilde{Q}^\dagger$ form a doublet. Though the $SU(2)_R$ action cannot be made manifest in terms of $N{=}1$ superfields, their transformations under an R-symmetry $U(1)_J \subset SU(2)_R$ are manifest in (2.1).

When $n_f{<}2n_c$, the theory is asymptotically free and generates a strong-coupling scale Λ, the instanton factor is proportional to $\Lambda^{2n_c - n_f}$, and the $U(1)_R$ symmetry is anomalous, being broken by instantons down to a discrete $\mathbb{Z}_{2n_c - n_f}$ symmetry. For $n_f = 2n_c$ the theory is scale-invariant and the $U(1)_R$ is not anomalous. In this case no strong coupling scale is generated, and the theory is described in terms of its bare coupling $\tau = \theta/\pi + i8\pi/g^2$.

We describe the selection rules resulting from the breaking of the classical symmetries by mass terms and instanton corrections by assigning symmetry transformation properties to the corresponding parameters in the action. In particular, the trace of the mass matrix m is a flavor singlet, m_S, while the traceless part transforms in an adjoint flavor representation, m_A.

The symmetry charges for the scalar component fields and parameters are thus

	$SU(n_c)$	$\times$	$SU(n_f)$	$\times$	$U(1)_B$	$\times$	$U(1)_R$	$\times$	$U(1)_J$	
Q	$\mathbf{n_c}$		$\mathbf{n_f}$		1		0		1	
$\widetilde{Q}$	$\mathbf{\overline{n}_c}$		$\mathbf{\overline{n}_f}$		-1		0		1	
Φ	$\mathbf{adj}$		$\mathbf{1}$		0		2		0	(2.4)
m_A	$\mathbf{1}$		$\mathbf{adj}$		0		2		0	
m_S	$\mathbf{1}$		$\mathbf{1}$		0		2		0	
$\Lambda^{2n_c - n_f}$	$\mathbf{1}$		$\mathbf{1}$		0		$2(2n_c - n_f)$		0	

The classical vacua are the zeroes of the scalar potential $V = \frac{1}{2}\mathrm{Tr}(D^2) + F^{i\dagger}F_i$ where $F_i = \partial \mathcal{W}/\partial \varphi^i$, $D^a = \varphi_i^\dagger (T^a)^i_j \varphi^j$, φ^i runs over all the (dynamical) scalar fields, and T^a are the gauge group generators in the representation carried by the φ fields. The classical vacua are thus found by setting the F and D terms to zero. The D-term equations are

$$0 = [\Phi, \Phi^\dagger],$$
$$\nu\delta^b_a = Q^i_a (Q^\dagger)^b_i - (\widetilde{Q}^\dagger)^i_a \widetilde{Q}^b_i, \qquad (2.5)$$

and the F-term equations are

$$\rho\delta^b_a = Q^i_a \widetilde{Q}^b_i,$$
$$0 = Q^j_a m^i_j + \Phi^b_a Q^i_b, \qquad (2.6)$$
$$0 = m^j_i \widetilde{Q}^a_j + \widetilde{Q}^b_i \Phi^a_b.$$

Here ν and ρ are arbitrary real and complex numbers, respectively. These vacuum equations imply that Φ, Q and $\widetilde{Q}$ may get vevs, which we denote by the same symbols. From the $N{=}1$ point of view, (2.5) come from the single D-term equation. The fact that both equations hold separately can be seen either by explicitly squaring the D-term and showing that the cross-terms cancel; or by noting that the first equation is an $SU(2)_R$ singlet while the second is part of a triplet (with the first equation in (2.6) and its adjoint) so must vanish separately; or, most generally, by noting that imposing the D-term equation and modding-out by gauge transformations is equivalent to dividing out by *complex* gauge transformations which can be used to diagonalize Φ, thus satisfying the first equation in (2.5) automatically.

We now study the solutions to the vacuum equations (2.5) and (2.6). The solutions fall into various "branches" corresponding to different phases of the theory. The Coulomb

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phase is defined as the set of solutions with $Q = \widetilde{Q} = 0$, the (pure) Higgs phase are solutions with $\Phi = 0$, and the mixed phases are those with both nonvanishing Φ and Q. For simplicity we will for the most part take vanishing bare masses $m_i^j = 0$.

2.2. Coulomb Branch

In the Coulomb phase Φ satisfies $[\Phi, \Phi^\dagger] = 0$ with $Q = \widetilde{Q} = 0$, implying that Φ can be diagonalized by a color rotation to a complex traceless matrix

$$\Phi = \text{diag}(\phi_1, \ldots, \phi_{n_c}), \qquad \sum_a \phi_a = 0. \qquad (2.7)$$

This vev generically breaks the gauge symmetry as $SU(n_c) \to U(1)^{n_c-1}$, motivating the name for this branch.

Gauge transformations in the Weyl group of $SU(n_c)$ act on the ϕ_a's by permuting them, so the Coulomb phase is described by identifying the n_c-1 complex-dimensional space of ϕ_a's under permutations. Gauge-invariant coordinates describing the Coulomb branch can be taken to be a basis of symmetric polynomials in the ϕ_a. This classical moduli space has orbifold singularities along submanifolds where some of the ϕ_a's are equal. In this case some of the non-Abelian gauge symmetry is restored. The scalar potential also gives the masses of the Q_a^i and $\widetilde{Q}_i^a$ squarks as $\phi_a + m_i$. The vanishing of one of these masses thus describes a complex codimension one submanifold in the Coulomb phase.

2.3. Higgs Branches

The Higgs branch is the space of solutions to the second equation in (2.5) and the first in (2.6) since $\Phi = 0$. Describe the squark fields as $n_c \times n_f$ complex matrices

$$Q = \begin{pmatrix} Q_1^1 & \cdots & Q_1^{n_f} \\ \vdots & & \vdots \\ Q_{n_c}^1 & \cdots & Q_{n_c}^{n_f} \end{pmatrix}, \qquad {}^t\widetilde{Q} = \begin{pmatrix} \widetilde{Q}_1^1 & \cdots & \widetilde{Q}_{n_f}^1 \\ \vdots & & \vdots \\ \widetilde{Q}_1^{n_c} & \cdots & \widetilde{Q}_{n_f}^{n_c} \end{pmatrix}, \qquad (2.8)$$

where ${}^t\widetilde{Q}$ denotes the transpose of $\widetilde{Q}$.

2.3.1 Solutions for the Squark Vevs

• *For $n_f \geq 2n_c$* any solution of the Higgs branch equations can be put in the following form by a combination of flavor and gauge rotations:

$$
Q = \begin{pmatrix} \kappa_1 & & & 0 & & & 0 & \\ & \ddots & & & \ddots & & & \ddots \\ & & \kappa_{n_c} & & & 0 & & \end{pmatrix}, \qquad \kappa_a \in \mathbb{R}^+,
$$

$$
{}^t\widetilde{Q} = \begin{pmatrix} \widetilde{\kappa}_1 & & & \lambda_1 & & & 0 & \\ & \ddots & & & \ddots & & & \ddots \\ & & \widetilde{\kappa}_{n_c} & & & \lambda_{n_c} & & \end{pmatrix}, \qquad \lambda_a \in \mathbb{R}^+,
$$

(2.9)

where

$$
\kappa_a \widetilde{\kappa}_a = \rho, \qquad \text{independent of } a, \qquad \rho \in \mathbb{C},
$$

$$
\lambda_a^2 = \kappa_a^2 - \frac{|\rho|^2}{\kappa_a^2} + \nu, \qquad \nu \in \mathbb{R}.
$$

(2.10)

We implicitly assumed above that the κ_a were all non-zero. If some of the κ_a vanish, then the solution (2.9) and (2.10) is still valid with the proviso that one sets $\rho = 0$ before setting any of the κ_a's to zero.

• *For $n_f < 2n_c$* solutions can be generated from $n_f = 2n_c$ solutions having corresponding flavor columns of Q and ${}^t\widetilde{Q}$ vanishing, by simply removing the columns in question. Conversely, to any solution of the Higgs branch equations for $n_f < 2n_c$ one can always add columns of zeros to get a solution with $n_f = 2n_c$, and the flavor rotations necessary to transform the solution into the form (2.9) can trivially be chosen not to act on these extra columns of zeros. Thus we are assured that this column-reduction procedure will generate from the $n_f = 2n_c$ solutions a solution in every flavor orbit of the $n_f < 2n_c$ solutions.

The only way one can reduce (2.9) by one or more columns is by setting $\lambda_1 = \cdots = \lambda_i = \kappa_1 = \cdots = \kappa_j = 0$ with $i+j = 2n_c - n_f$. We will find in this way two separate classes of solutions which we term the baryonic and non-baryonic branches. The motivation for this terminology will become clear shortly.

The Baryonic Branch. Choosing $i = 2n_c - n_f$ and $j = 0$, we find the squark vevs

$$
Q = \begin{pmatrix} \kappa_1 & & & & & & \\ & \ddots & & & & & \\ & & \kappa_{n_f - n_c} & & & & \\ & & & \kappa_0 & & & \\ & & & & \ddots & & \\ & & & & & \kappa_0 & \end{pmatrix}, \qquad \kappa_a \in \mathbb{R}^+,
$$

$$
{}^t\widetilde{Q} = \begin{pmatrix} \widetilde{\kappa}_1 & & & & & & & \lambda_1 & & \\ & \ddots & & & & & & & \ddots & \\ & & \widetilde{\kappa}_{n_f - n_c} & & & & & & & \lambda_{n_f - n_c} \\ & & & \widetilde{\kappa}_0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \widetilde{\kappa}_0 & & & & \end{pmatrix}, \lambda_a \in \mathbb{R}^+,
$$

(2.11)

where

$$
\kappa_a \widetilde{\kappa}_a = \rho, \qquad \text{independent of } a, \qquad \rho \in \mathbb{C},
$$

$$
\lambda_a^2 = \kappa_a^2 - \kappa_0^2 + |\rho|^2 \left(\frac{1}{\kappa_a^2} - \frac{1}{\kappa_0^2} \right).
$$

(2.12)

We will apply the term baryonic branch to the solutions (2.9) for $n_f \geq 2n_c$ as well. Note that the baryonic branch exists for $n_f \geq n_c$. It is not hard to see that reducing (2.9) as above but with $i = 0$ and $j = 2n_c - n_f$ leads to a submanifold of the same branch upon interchanging Q with $\widetilde{Q}$, which is a symmetry (charge conjugation) of our theory.[3]

The Non-Baryonic Branches. Another possibility for column-reduction of (2.9) is to have some of both the κ_a's and the λ_a's vanish. However, from the various constraints in (2.9) and (2.10), it follows that $\rho = \nu = 0$ and $\kappa_a = \lambda_a$. Thus the squark vevs on this

---

[3] It is not *a priori* clear that this symmetry relates points on the same Higgs branch rather than on two isomorphic Higgs branches. This follows from the definition, given below, of what it means for Higgs branches to be separate.

branch are parametrized as

$$
Q = \begin{pmatrix} \kappa_1 & & & 0 & & & 0 & & \\ & \ddots & & & \ddots & & & \ddots & \\ & & \kappa_r & & & 0 & & & \\ & & & & & & & & \end{pmatrix},
$$

$$
{}^t\widetilde{Q} = \begin{pmatrix} 0 & & & \kappa_1 & & & 0 & & \\ & \ddots & & & \ddots & & & \ddots & \\ & & 0 & & & \kappa_r & & & \\ & & & & & & & & \end{pmatrix}, \qquad \kappa_a \in \mathbb{R}^+,
$$

$$(2.13)$$

where $r \leq [n_f/2]$ and square brackets denote integer part. Also, $2n_c - n_f$ columns of zeros should be deleted by the column reduction procedure. Note that when n_f is odd, there remains at least one column of zeros in the reduced matrices. We will refer to the manifolds obtained for different values of r as distinct non-baryonic branches despite the fact that in (2.13) they appear as submanifolds of the branch with the maximal value of r by setting some of the $\kappa_a = 0$. Also, some of the non-baryonic branches are obtained as submanifolds of the baryonic branch by setting $\rho = \kappa_0 = \widetilde{\kappa}_0 = 0$ in (2.11). The reason for these distinctions will become clearer below. Non-baryonic branches exist for $n_f \geq 2$. For $n_f < 2$ there are no Higgs branches.

2.3.2 Gauge Symmetry and Separate Branches

To make the pattern of intersections of Higgs branches clearer, we define two Higgs branches to be separate if every path between the two goes through a point of enhanced gauge symmetry. In particular, if one branch has a larger unbroken gauge group than another, they are separate branches. This will become clear in subsection 2.4 when we discuss the mixed branches.

The Baryonic Branch. The generic solution (2.9) or (2.11) completely breaks the gauge symmetry. By the Higgs mechanism, the number of massless hypermultiplets is $\mathcal{H} = n_f n_c - n_c^2 + 1$, counting the quaternionic dimension of the Higgs branch. There are submanifolds of the baryonic branch where the gauge symmetry is enhanced. These occur when two or more rows of Q and $\widetilde{Q}$ vanish. This can only happen if $\rho = \nu = 0$ in (2.9) or

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$\rho = \kappa_0 = 0$ in (2.11), giving rise to non-baryonic branch vevs (2.13) with

$$r \leq \min\{\, n_f - n_c\, ,\ n_c - 2\,\}. \tag{2.14}$$

The Non-Baryonic Branches. For $n_f < 2n_c$ there are nonbaryonic branches with r outside the range (2.14). In general on the nonbaryonic branch the unbroken gauge group is then $SU(n_c - r)$ with $n_f - 2r$ massless hypermultiplets in the fundamental. Since there are different unbroken gauge symmetries for each value of r they are separate branches. (Even though the non-baryonic branch with $r = n_c - 1$ does not have an unbroken gauge symmetry, it is easy to see that it is still separate from the baryonic branch: any path connecting the two must go through a point with an unbroken $SU(2)$ gauge symmetry.) The Higgs mechanism gives for the number of massless multiplets neutral under the unbroken gauge group $\mathcal{H} = r(n_f - r)$.

2.3.3 Flavor Symmetry

In order to identify the unbroken global symmetries on the Higgs branches, it is useful to define a basis of gauge-invariant quantities made from the squark vevs, the meson and baryon fields

$$M^i_j = \widetilde{Q}^a_j Q^i_a,$$

$$B^{i_1 \ldots i_{n_c}} = Q^{i_1}_{a_1} \cdots Q^{i_{n_c}}_{a_{n_c}}\, \epsilon^{a_1 \ldots a_{n_c}}\ , \tag{2.15}$$

$$\widetilde{B}_{i_1 \ldots i_{n_c}} = \widetilde{Q}^{a_1}_{i_1} \cdots \widetilde{Q}^{a_{n_c}}_{i_{n_c}}\, \epsilon_{a_1 \ldots a_{n_c}}\ .$$

The baryon fields are only defined for $n_f \geq n_c$.

The Baryonic Branch. On this branch $B, \widetilde{B} \neq 0$, hence its name. From (2.9) or (2.11) the meson field M^i_j is

$$M = \begin{pmatrix} \rho & & & \kappa_1 \lambda_1 & & & 0 & \\ & \ddots & & & \ddots & & & \ddots \\ & & \rho & & & \kappa_{n_c} \lambda_{n_c} & & \\ & & & & & & & \\ & & & & & & & \end{pmatrix}, \tag{2.16}$$

where the ρ-block is $n_c \times n_c$. For $n_f < 2n_c$ one should remove the appropriate number of columns from the right and rows from the bottom of (2.16).

• *For $n_f \geq 2n_c$*, (2.16) and the non-vanishing baryon vevs imply the global symmetry is broken as[4] $SU(n_f) \times U(1)_B \times SU(2)_R \to U(n_f - 2n_c) \times U(1)^{n_c - 1} \times SU(2)'_R$. (For $n_f = 2n_c$ drop the $U(0)$ factor.) Thus the number of (real) Goldstone bosons is $\mathcal{G} = 4n_f n_c - 4n_c^2 - n_c + 1$. Since the number of (real) parameters describing the Higgs branch in (2.9) is $\mathcal{P} = n_c + 3$, the fact that $\mathcal{G} + \mathcal{P} = 4\mathcal{H}$ is a check that we have a complete parametrization of this branch.

• *For $n_c \leq n_f < 2n_c$* the global symmetry is broken as $SU(n_f) \times U(1)_B \times SU(2)_R \to SU(2n_c - n_f) \times U(1)^{n_f - n_c} \times SU(2)'_R$. (For $n_f = 2n_c - 1$ drop the $SU(1)$ factor.) Thus the number of Goldstone bosons is $\mathcal{G} = -4n_c^2 + 4n_c n_f - n_f + n_c + 1$, the number of parameters describing the baryonic branch is $\mathcal{P} = n_f - n_c + 3$, and $\mathcal{G} + \mathcal{P} = 4\mathcal{H}$.

The Non-Baryonic Branches. The baryon fields vanish on these branches, $B = \widetilde{B} = 0$, hence their name. The meson field on these branches is given by

$$M = \begin{pmatrix} 0 & & \kappa_1^2 & & 0 & \\ & \ddots & & \ddots & & \ddots \\ & & 0 & & \kappa_r^2 & \\ & & & & & \\ & & & & & \end{pmatrix}, \tag{2.17}$$

where the first block of zeros is $r \times r$. This implies the global symmetry is broken as $SU(n_f) \times U(1)_B \times SU(2)_R \to U(n_f - 2r) \times U(1)^r \times SU(2)'_R$. Hence $\mathcal{G} = r(4n_f - 4r - 1)$, while $\mathcal{P} = r$ and $\mathcal{G} + \mathcal{P} = 4\mathcal{H}$.

2.3.4 <u>Gauge Invariant Description of the Higgs Branches</u>

Eq. (2.9) only gives representative solutions for the squark vevs in the pure Higgs phases. Global symmetry transformations on these solutions relate them to inequivalent points in the moduli space (with identical physics). Gauge transformations relate them to equivalent points. Therefore, it is useful to describe the moduli space in terms of gauge invariant coordinates. Such a description will prove useful in section 4. So, we would like to describe the various branches in terms of constraints on the gauge-invariant mesons and baryons (2.15).

---

[4] The meson vev by itself only breaks it to $U(1)^{n_c}$, but the non-vanishing baryon vevs break baryon number as well.

Mathematically, the Higgs branch is a hyperKähler quotient of squark space by the gauge group, with the D- and F-terms as moment maps. We find it useful to work instead with a Kähler quotient, essentially considering the theory as an $N=1$ model with a super-potential interaction. In a Kähler quotient, setting the D-terms to zero and identifying orbits of the gauge group is equivalent to a quotient by the complexified gauge group. We achieve this by expressing the vevs directly in terms of holomorphic gauge-invariant coordinates, the mesons and baryons (2.15), and imposing the F-term equations. The nontrivial structure of the quotient (or the D-term equations) is manifested in the fact that the gauge-invariant coordinates are not independent as functions of the squarks but satisfy a set of polynomial relations which we must now impose as constraints. Below we find a set of generators for the constraints following from these relations (to which we refer somewhat loosely as D-terms) and the F-term equations.

Since the product of two color epsilon-tensors is the antisymmetrized sum of Kronecker deltas, it follows that

$$B^{i_1 \ldots i_{n_c}} \widetilde{B}_{j_1 \ldots j_{n_c}} = M^{[i_1}_{j_1} \cdots M^{i_{n_c}]}_{j_{n_c}}, \tag{2.18}$$

where the square brackets imply antisymmetrization. Also, since any expression antisymmetrized on n_c+1 color indices must vanish, it follows that any product of M's, B's, and $\widetilde{B}$'s antisymmetrized on n_c+1 upper or lower flavor indices must vanish. As long as both B and $\widetilde{B}$ are non-zero, an induction argument shows that the two constraints

$$(*B)\widetilde{B} = *(M^{n_c}), \tag{2.19}$$

$$M \cdot *B = M \cdot *\widetilde{B} = 0, \tag{2.20}$$

imply all the other D-term constraints.[5] Here the "$\cdot$" represents the contraction of an upper with a lower flavor index, and the "$*$" stands for contracting all flavor indices with the totally antisymmetric tensor on n_f indices. For example

$$(*B)_{i_{n_c+1} \ldots i_{n_f}} = \epsilon_{i_1 \ldots i_{n_f}} B^{i_1 \ldots i_{n_c}}. \tag{2.21}$$

---

[5] A constraint with, say, k B's and an arbitrary number of M's antisymmetrized on n_c+1 upper indices can be replaced by a constraint with $k-1$ B fields by (2.19). Repeating this process reduces all constraints to (2.19) plus the single constraint with no B fields $*(M^{n_c+1}) = 0$. We show below that this latter constraint is implied by (2.20).

Note that (2.19) is just (2.18) rewritten in this notation. If $B = \widetilde{B} = 0$, then all the other constraints are automatically satisfied (being proportional to B or $\widetilde{B}$) and (2.19) implies (2.20).[6] It is useful to note that (2.19) and (2.20) imply $0 = \widetilde{B}(M \cdot *B) = M \cdot *(M^{n_c}) = *(M^{n_c+1})$, implying

$$\text{rank}(M) \leq n_c. \tag{2.22}$$

The F-term equations (2.6) imply further constraints. The first such equation[7] implies

$$M' \cdot B = \widetilde{B} \cdot M' = 0, \tag{2.23}$$

$$M \cdot M' = 0, \tag{2.24}$$

as well as another constraint which is just a contraction of (2.19). In the above we have defined

$$(M')^i_j \equiv M^i_j - \frac{1}{n_c}(\text{Tr}M)\delta^i_j. \tag{2.25}$$

Thus (2.19), (2.20), (2.23), and (2.24) are a complete set of constraints.

The F-term condition (2.24) by itself is quite restrictive. Its only solutions, up to flavor rotations, are in fact precisely (2.16) and (2.17). To see this, assume that $\text{rank}(M) = r$. By an $SU(n_f)$ similarity transformation M can be put into the form

$$M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \tag{2.26}$$

where A is an $r \times r$ block and B is an $r \times (n_f - r)$ block. Since $\text{rank}(M) = r$, the $r \times n_f$ block $(\, A \; B \,)$ must have r linearly independent columns. (2.24) gives $(A - \frac{1}{n_c}\text{Tr}A) \cdot (\, A \; B \,) = 0$, implying

$$A^i_j = \frac{1}{n_c}(\text{Tr}A)\delta^i_j. \tag{2.27}$$

The solutions of this equation are either A is diagonal and $r = n_c$, or $A = 0$ and there is no restriction on r. An $SU(n_f)$ similarity transformation which preserves the form (2.26)

---

[6] When only one of B or $\widetilde{B}$ vanishes extra constraints beyond (2.19) and (2.20) are required. However, such Higgs branches are only submanifolds of the baryonic branch, not separate branches. This follows from our explicit parametrization of the squark vevs (2.9). Thus we can safely ignore this special case for the purposes of identifying and counting Higgs branches.

[7] The other two equations are relevant only for a description of mixed branches.

of M can be chosen to diagonalize B. Thus the solutions with $r = n_c$ are seen to coincide with the baryonic branch solutions (2.16), while the solutions with $A = 0$ are of the form (2.17) found for the non-baryonic branch. Note that the non-baryonic solutions have rank $r \leq [n_f/2]$. For $n_f > 2n_c$, this will be reduced to $r \leq n_c$ by (2.22). For $n_f \leq 2n_c$, on the other hand, this constraint is automatically satisfied, and (2.22) is implied by (2.24).

2.4. Mixed Branches

So far in our analysis we have not taken the last two equations in (2.6) into account. They have no effect on the Coulomb branch, and on the Higgs branches they are clearly satisfied for vanishing masses. When the masses do not vanish they put constraints on the Higgs vevs. Intuitively this is clear: the Higgs phase corresponds to flat directions along which some components of the quark superfields remain massless, so bare mass terms tend to lift these flat directions. Indeed, there are no non-zero masses satisfying the constraint (2.3) for which the generic Higgs branch (2.9) for $n_f = 2n_c$ satisfies the last two equations in (2.6).

These equations play a more interesting role on mixed branches. By mixed branches, we mean solutions to the vacuum equations for which both Q or $\widetilde{Q}$ and Φ are non-zero. For simplicity assume that the bare masses vanish. The Φ vev can be put in the diagonal form (2.7) by a color transformation. Then the last two equations in (2.6) only have a non-zero solution for Q, $\widetilde{Q}$, and Φ if the squark and adjoint-scalar vevs live in disjoint color subgroups. This leads to a clean distinction between the various Higgs branches. Branches with different gauge groups are distinct because they appear as the Higgs factor of mixed branches with manifestly distinct Coulomb factors.

This fact and the other equations in (2.5) and (2.6) imply that the vevs can be parametrized up to gauge and flavor rotations as

$$\Phi = \text{diag}(0, \ldots, 0, \phi_{r+1}, \ldots, \phi_{n_c}), \qquad \phi_a \in \mathbb{C}, \qquad \sum_a \phi_a = 0, \qquad (2.28)$$

and as in (2.13) for the squarks. (The squark vevs for $n_f < 2n_c$ are obtained by column reduction of (2.13).) Thus locally the mixed branch has the structure of a direct product of a non-baryonic Higgs branch and a Coulomb branch. This Coulomb branch can be

identified with the Coulomb branch of the unbroken $SU(n_c-r)$ gauge theory of the non-baryonic Higgs branch. Henceforth, we will refer to the mixed branches as non-baryonic branches.

2.5. Summary of Classical Moduli Space

This long section has been devoted to solving the classical vacuum equations of $N{=}2$ supersymmetric $SU(n_c)$ with n_f massless fundamental flavors. We can summarize what we have learned by recording the number $\mathcal{V}$ of massless $U(1)$ vector multiplets and $\mathcal{H}$ of massless neutral hypermultiplets at a generic point on the various branches of the moduli space:

- The baryonic branch exists for $n_f \geq n_c$ with $\mathcal{V} = 0$ and $\mathcal{H} = n_f n_c - n_c^2 + 1$.

- The non-baryonic branches exist for $n_f \geq 2$ with $\mathcal{V} = n_c{-}1{-}r$ and $\mathcal{H} = r(n_f{-}r)$, where $1 \leq r \leq \min\{\,[n_f/2]\,,\,n_c{-}2\,\}$. (Note that $[n_f/2]$ is the lesser when $n_f < 2n_c{-}2$.)

- The Coulomb branch exists for all n_f with $\mathcal{V} = n_c{-}1$ and $\mathcal{H} = 0$.

We have also learned how the Coulomb, baryonic and non-baryonic branches fit together. Classically, Higgs branches intersect the Coulomb branch at the origin, and out of the Higgs branch emanate various mixed branches which touch the Coulomb branch along submanifolds where two or more squarks are massless (which is where also, classically, a non-Abelian gauge group is unbroken). Thus, from (2.28), when the non-baryonic branches meet the Coulomb branch there is classically an $SU(r) \times U(1)^{n_c-r}$ unbroken gauge group with n_f massless hypermultiplets in the fundamental representation of $SU(r)$ and charged under one of the $U(1)$ factors (by an appropriate choice of basis of the $U(1)$'s). Similarly, the baryonic branch meets the Coulomb branch at its origin where, classically, the full $SU(n_c)$ is unbroken with n_f massless fundamental flavors. Finally, the various Higgs branches also connect up along submanifolds of enhanced gauge symmetry. In particular, the baryonic branch (with no gauge symmetry generically) meets the non-baryonic branches on a submanifold with enhanced gauge group $SU(n_c{-}r)$ for $r \leq \min\{\,n_f{-}n_c\,,\,n_c{-}2\,\}$. This classical picture is sketched in Fig. 1, where only one non-baryonic branch is depicted.

3. Quantum Higgs Branches and the Non-Renormalization Theorem

So far our analysis has been completely classical. The question remains how this structure is modified quantum-mechanically. We will now show that there are no quantum corrections to the baryonic and non-baryonic branches, though where they intersect each other and the Coulomb branch may be modified.

We have seen above that the classical moduli space is constructed from various branches, and the generic point on a given branch is an Abelian theory with some numbers $\mathcal{V}$ of vector multiplets and $\mathcal{H}$ of neutral hypermultiplets. Far enough out along any of theses branches the physics is weakly coupled if the microscopic theory is asymptotically free. (In the scale-invariant case, $n_f = 2n_c$, we can make the physics weakly coupled by taking the bare coupling small.) Thus in these limits we expect our description of the low energy effective theory in terms of Abelian vector multiplets and neutral hypermultiplets to be accurate. We will now study how it can change.

Consider an arbitrary $N{=}2$ supersymmetric theory of Abelian vector multiplets and neutral hypermultiplets. It was shown in [20] that the scalar components of vector multiplets cannot appear in the hypermultiplet metric, and *vice versa*. For global supersymmetry one can easily derive that as follows. Say we have neutral hypermultiplets with $N{=}1$ chiral multiplets Q^i, $\widetilde{Q}^i$, $i{=}1,\ldots,\mathcal{H}$ and Abelian vector multiplets containing the $N{=}1$ chiral multiplets Φ^a, $a{=}1,\ldots,\mathcal{V}$. Denote their scalar components by q^i, $\widetilde{q}^i$, and ϕ^a. $N = 1$ supersymmetry implies that the Kähler potential is some general function $K(Q^i, \widetilde{Q}^i, \Phi^a, Q^{i+}, \widetilde{Q}^{i+}, \Phi^{a+})$. Then the kinetic terms for the scalars will include the cross terms $\partial_i \partial_{\overline{a}} K \cdot \partial_\mu q^i \partial^\mu \phi^{a+} + c.c.$. By $N{=}2$ supersymmetry such a term must be accompanied by a term involving $\partial_i \partial_{\overline{a}} K \cdot \overline{\psi}_{\widetilde{q}^i} \slashed{\partial} \lambda^a$, where $\psi_{\widetilde{q}}$ is the hypermultiplet fermion and λ the gaugino. But such a term is not allowed since to cancel its ($N{=}1$) supersymmetry variation one needs a term involving two derivatives and A^a_μ as well as scalars, out of which no Lorentz-invariant combination can be formed. Therefore $\partial_i \partial_{\overline{a}} K = 0$, implying $K = K_H(Q^i, \widetilde{Q}^i, Q^{i+}, \widetilde{Q}^{i+}) + K_V(\Phi^a, \Phi^{a+})$.

This result implies that quantum mechanically the various branches of the moduli space retain their local product structure (*e.g.* the "mixed" non-baryonic branch is still locally a product of a Coulomb branch and a Higgs branch).

Further constraints on the quantum theory can be deduced by extending the non-renormalization theorem of [21] to $N = 2$ supersymmetry. We can promote all the parameters in the theory to background $N = 2$ superfields. The way the parameters appear in the Lagrangian constrain the supersymmetry representation they belong to. For example, after rescaling Q and $\widetilde{Q}$ in (2.1) by $\sqrt{\tau}$ the gauge coupling constant τ in (2.1) multiplies the classical prepotential. Hence it can be promoted to a background vector superfield. In the quantum theory τ is replaced by the dynamically generated scale Λ ($\tau \sim \log \Lambda$). Therefore, we can think of $\log \Lambda$ as a background $U(1)$ vector superfield. Since the metric on the Higgs branch is independent of vector superfields, it is independent of Λ. Finally, we can use the fact that the classical theory is obtained in the limit $\Lambda \to 0$ to conclude that that metric is given exactly by the classical answer.[8]

It is also useful to promote the quark masses m_j^i to background superfields. Their couplings in the superpotential (2.2) are identical to those of the scalar component of a vector superfield (compare the first and the second term in (2.2)). These vector superfields correspond to the gauging of the global $SU(n_f) \times U(1)_B$ flavor symmetry. m_A corresponds to the vectors of $SU(n_f)$ while m_S is associated with $U(1)_B$. Using this point of view the restriction (2.3) follows from the D-term equations for these superfields. As in the previous paragraph, we immediately learn that the metric on the Higgs branch is independent of the masses.

We thus learn that only the Coulomb branch can receive quantum corrections, and that the mixed non-baryonic branch will retain its classical product structure of a hypermultiplet manifold times the vector multiplet manifold corresponding to the subspace of the Coulomb branch along which the non-baryonic and Coulomb branches intersect. Though we have not used this fact, the $N{=}2$ supersymmetry implies that the hypermultiplet and vector multiplet manifolds are further constrained to be hyperKähler and rigid special Kähler manifolds, respectively. The rigid special Kähler structure has been used to find the exact Coulomb branch, and this solution will be reviewed in section 5. One result

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[8] A similar argument was recently used in string theory [22] to derive a similar non-renormalization theorem. There the coupling constant is always a dynamical field – the dilaton – and one does not need to promote a parameter to a background field.

we will need now, though, is that globally the Coulomb branch can be characterized by n_c complex numbers ϕ_a (up to permutations) whose sum vanishes, just as in the classical analysis of the last section. These coordinates ϕ_a can be viewed as the eigenvalues of the adjoint scalar Φ vev.

4. Higgs Branch Roots

In this and the next section we identify the roots of the baryonic and non-baryonic branches (*i.e.* the submanifolds where they intersect the Coulomb branch) in the quantum theory. We will, from now on, concentrate on the asymptotically free and finite theories with $n_f \leq 2n_c$. In this section we will identify the physics of the roots using simple physical arguments. The upshot of this analysis will be that the roots of the non-baryonic branches are where the gauge symmetry is enhanced to $SU(r) \times U(1)^{n_c-r}$ with n_f flavors, just as in the classical analysis. The root of the baryonic branch, though, will be shown to have an unbroken gauge symmetry $SU(n_f-n_c) \times U(1)^{2n_c-n_f}$ with n_f flavors and some singlets. Note that these are all non-asymptotically-free theories, and so are weakly coupled in the IR. In section 5 we use this information to precisely locate the roots using the exact solution for the Coulomb branch.

4.1. The Non-Baryonic Root

In section 2 we identified the non-baryonic root as the submanifold of the Coulomb branch with unbroken $SU(r) \times U(1)^{n_c-r}$ gauge symmetry for $r \leq [n_f/2]$, with n_f massless fundamental hypermultiplets charged under a single $U(1)$ factor. When $r < [n_f/2]$, or $r = [n_f/2]$ and n_f is odd, this gauge theory is IR–free, and thus we expect that the non-Abelian factor will not be broken in the infrared by quantum-mechanical effects. When $r = [n_f/2]$ and n_f is even, the non-Abelian factor is scale invariant. From the exact solution for the low-energy effective action on the Coulomb branch of such theories it is known that a vacuum with unbroken $SU(n_f/2)$ gauge symmetry survives in the infrared (this is shown in section 5). We thus conclude that even with quantum-mechanical corrections, the non-baryonic root will remain the submanifold of unbroken gauge group determined classically.

We can make a few checks on this conclusion. Since the theory at the root is weakly coupled, we can use it to compute the Higgs branch emanating from it. By the nonrenormalization theorem the result should be the same as the classical computation of the non-baryonic branches performed in section 2. Recall that that computation found the non-baryonic branch has $\mathcal{V} = n_c - r - 1$ massless $U(1)$ vector multiplets. These clearly correspond to the $U(1)$ factors of the IR theory under which none of the hypermultiplets are charged. In section 2 we also learned that the non-baryonic branch has $\mathcal{H} = r(n_f - r)$ massless neutral hypermultiplets, an unbroken global symmetry group of $U(n_f - 2r) \times U(1)^r \times SU(2)_R$, and can be described by squark vevs of the form (2.13). These properties can be easily recovered from the IR theory. The relevant gauge group (under which the hypermultiplets are charged) is $SU(r) \times U(1)$, so the analysis of the Higgs branch is similar to that of section 2 with the number of colors now given by r. The condition coming from the $U(1)$ factors amounts to setting $\nu = \rho = 0$ in (2.5) and (2.6). The $n_f \geq 2n_c$ squark vevs (2.9) then indeed take the form (2.13), and it follows that the number of hypermultiplets and global symmetries match those of the non-baryonic branch.

Along the root of the non-baryonic branch, though generically there is no light matter charged under the $n_c - r - 1$ $U(1)$'s, it is found from the exact solution for the Coulomb branch that there are special submanifolds where some (monopole) hypermultiplets charged with respect to these $U(1)$'s become massless. Such special vacua will be important in breaking to $N{=}1$ supersymmetry (section 6) which lifts all the vacua in the non-baryonic root except for those with a massless charged hypermultiplet for each $U(1)$ factor. Such vacua occur at isolated points on the baryonic root—we will determine their coordinates in section 5 using the exact solution on the Coulomb branch. We will find that, by a combination of electric-magnetic duality rotations in the $U(1)$ factors and an appropriate change of basis among the $U(1)$'s, we can take the singlet hypermultiplets to each have charge 1 under a single $U(1)$:

$$
\begin{array}{c|ccccccc}
 & SU(r) & \times\; U(1)_0 & \times\; U(1)_1 & \times\; \cdots & \times\; U(1)_{n_c-r-1} & \times\; U(1)_B \\
n_f \times q & \mathbf{r} & 1 & 0 & \cdots & 0 & 0 \\
e_1 & \mathbf{1} & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
e_{n_c-r-1} & \mathbf{1} & 0 & 0 & \cdots & 1 & 0
\end{array}
\tag{4.1}
$$

Here q^i are the n_f quark multiplets, and the e_k label the n_c-r-1 singlet hypermultiplets at such a special point. We have included the $U(1)_B$ global baryon numbers. Since we are free to redefine a global charge by adding arbitrary multiples of any local charges, it is clear from (4.1) that we are able to set the baryon number of all the fields to zero as shown. We find $2n_c-n_f$ such vacua, related by a $\mathbb{Z}_{2n_c-n_f}$ discrete symmetry on the Coulomb branch (the anomaly-free subgroup of the classical $U(1)_R$ symmetry) which leaves the non-baryonic root invariant. (The case $r = n_f - n_c$ is an exception to this last statement.)

4.2. The Baryonic Root

Locating the baryonic root is more difficult. The reason is that classically the baryonic root is the origin of the Coulomb branch, where the full $SU(n_c)$ gauge group is unbroken. But, for $n_f < 2n_c$ this vacuum is asymptotically-free (AF), and thus will be altered in the IR by strong quantum effects. Indeed, in the exact solution for the Coulomb branch, it is found that the singularity corresponding to the classical vacuum does not appear, and instead is "split" into various singularities all at distances $\sim \Lambda$ from the origin. Here Λ is the dynamically generated strong-coupling scale of the AF theory.

The one exception to this strong-coupling difficulty is the scale-invariant case, when $n_f = 2n_c$, and the bare coupling, τ, is a parameter of the theory. At weak coupling, $\tau \to +i\infty$, we know that the vacuum in question is at the origin of the Coulomb branch. (Using the exact solution, we learn in section 5 that it remains there even at strong coupling.)

For $n_f < 2n_c$, the $U(1)_R$ symmetry of the classical theory is broken by anomalies to a discrete $\mathbb{Z}_{2n_c-n_f}$ symmetry. This acts on the Φ vev by multiplication by a phase. Since the baryonic root is a single point (a fact which cannot be modified quantum-mechanically by the non-renormalization theorem) it must be invariant under the $\mathbb{Z}_{2n_c-n_f}$ symmetry. For $n_f \geq n_c$ the most general diagonal Φ vev (coordinates on the Coulomb branch) which are invariant are of the form[9]

$$\Phi \propto (\underbrace{0,\ldots,0}_{n_f-n_c},\omega,\omega^2,\ldots,\omega^{2n_c-n_f}), \tag{4.2}$$

---

[9] Larger-dimensional submanifolds with this property exist for $n_f \geq \frac{3}{2}n_c$: The k-complex

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where $\omega = \exp\{2\pi i/(2n_c - n_f)\}$. Note that, up to overall normalization, there is only one such point since the components of Φ are identified up to permutations. Points on the submanifold (4.2) classically have unbroken gauge group $SU(n_f - n_c) \times U(1)^{2n_c - n_f}$ with n_f fundamental hypermultiplets. Since this is an IR–free theory, we expect its gauge symmetry to survive quantum-mechanically. Quantum effects could, however, change this effective theory by bringing down additional light degrees of freedom. In particular, there may be points on the submanifold (4.2) where (monopole) singlets charged under the $U(1)$ factors become massless. It is straight-forward to show that such a theory has no purely hypermultiplet Higgs branch (like the baryonic one) unless there is at least one singlet charged under each $U(1)$ factor. Let us assume there is such a point with precisely $2n_c - n_f$ singlets; in section 5 we will show that such a point exists and compute the normalization of Φ in (4.2) for this point using the exact solution.

If the effective theory at the baryonic root is $SU(n_f - n_c) \times U(1)^{2n_c - n_f}$ with n_f massless squarks q^i and $2n_c - n_f$ massless singlets e_k, we can pick a basis in which the singlets have a diagonal charge matrix, with each singlet having charge -1 under only one of the $U(1)$'s. We will show later that the squarks then have charge $1/(n_f - n_c)$ under each of the $U(1)$ factors.[10] These charges can be summarized as follows:

$$
\begin{array}{cccccccc}
 & SU(\widetilde{n}_c) & \times & U(1)_1 & \times & \cdots & \times & U(1)_{n_c - \widetilde{n}_c} & \times & U(1)_B \\
n_f \times q & \mathbf{\widetilde{n}_c} & & 1/\widetilde{n}_c & & \cdots & & 1/\widetilde{n}_c & & -n_c/\widetilde{n}_c \\
e_1 & \mathbf{1} & & -1 & & \cdots & & 0 & & 0 \\
\vdots & \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
e_{n_c - \widetilde{n}_c} & \mathbf{1} & & 0 & & \cdots & & -1 & & 0
\end{array}
\tag{4.3}
$$

where we have defined

$$
\widetilde{n}_c \equiv n_f - n_c, \tag{4.4}
$$

dimensional submanifold

$$
\Phi = (0, \ldots, 0, \underbrace{\varphi_1 \omega^i}_{2n_c - n_f}, \ldots, \underbrace{\varphi_k \omega^i}_{2n_c - n_f})
$$

exists for $n_f \geq (\frac{k+1}{2k})n_f$, along which the unbroken gauge group is $SU(kn_f - (2k-1)n_c) \times U(1)^{2kn_c - kn_f}$. It will turn out, however, that there are no points along this submanifold where $2kn_c - kn_f$ or more monopoles become massless; see section 5.

[10] This is determined later in this subsection by the requirement that the Higgs branch be correctly reproduced. In the next subsection we will rederive this result from a different argument.

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a combination which will appear frequently below. We have included the charges under the global $U(1)_B$ baryon number. An appropriate combination of any such $U(1)_B$ with the gauge $U(1)$'s will eliminate the baryon number of the singlets, as shown. The normalization of the baryon number will be determined below by matching to the elementary squarks Q whose baryon number is defined to be 1.

We now would like to show that the vacua (4.3) are indeed the root of the baryonic Higgs branch. First of all, this is plausible because the dimensions and global symmetries of the two Higgs branches are the same. To see this, write down the vacuum equations for the theory (4.3). Let ϕ be the adjoint scalar component of the vector multiplet for the $SU(\widetilde{n}_c)$ gauge group, and ψ_k, $k = 1, \ldots, n_c - \widetilde{n}_c$ be the adjoint scalars for the $U(1)$ multiplets. Then the superpotential for the theory is

$$\mathcal{W}/\sqrt{2} = \mathrm{tr}(\widetilde{q} \cdot q \phi) + \frac{1}{\widetilde{n}_c} \mathrm{tr}(q \cdot \widetilde{q}) \Big(\sum_k \psi_k \Big) - \sum_k \psi_k e_k \widetilde{e}_k, \tag{4.5}$$

where the trace is over the $\widetilde{n}_c$ color indices and the dot product is a sum over the n_f flavor indices. The resulting D-term equations are

$$|e_k|^2 - |\widetilde{e}_k|^2 = \nu,$$
$$q \cdot q^\dagger - \widetilde{q}^\dagger \cdot \widetilde{q} = \nu, \tag{4.6}$$
$$[\phi, \phi^\dagger] = 0,$$

and F-term equations are

$$e_k \widetilde{e}_k = \rho,$$
$$q \cdot \widetilde{q} = \rho,$$
$$\psi_k e_k = \psi_k \widetilde{e}_k = 0, \tag{4.7}$$
$$\Big(\widetilde{n}_c \phi + \sum_k \psi_k \Big) q = \widetilde{q} \Big(\widetilde{n}_c \phi + \sum_k \psi_k \Big) = 0,$$

with no summation implied over repeated indices, and with $\nu \in \mathbb{R}$, $\rho \in \mathbb{C}$. The ϕ and q equations are essentially the same as in (2.5) and (2.6). On the pure Higgs branch (*i.e.* setting $\psi_k = \phi = 0$) these equations determine e_k and $\widetilde{e}_k$ up to phases, and the remaining equations for q and $\widetilde{q}$ are the same as the Higgs branch equations solved in section 2.3.

Thus, the generic solution up to flavor rotations is given by (2.9) and (2.10) with n_c replaced by $\widetilde{n}_c$. This solution completely breaks the $SU(\widetilde{n}_c) \times U(1)^{n_c - \widetilde{n}_c}$ gauge symmetry. Since the total number of hypermultiplets is $n_f \widetilde{n}_c + n_c - \widetilde{n}_c$, by the Higgs mechanism the number of massless hypermultiplets remaining on this branch is $\mathcal{H} = n_f \widetilde{n}_c + n_c - \widetilde{n}_c - (\widetilde{n}_c^2 - 1 + n_c - \widetilde{n}_c) = n_f n_c - n_c^2 + 1$, matching that found in section 2.3 for the baryonic branch. Furthermore, it is easy to see that the global symmetry is also broken in the same way: $SU(n_f) \times U(1)_B \to U(n_f - 2\widetilde{n}_c) \times U(1)^{\widetilde{n}_c - 1} = SU(2n_c - n_f) \times U(1)^{\widetilde{n}_c}$.

By the non-renormalization theorem, the Higgs branch of the effective theory at the root of the baryonic branch (4.3) should be precisely the same as the baryonic branch of the electric theory. We will now show this by describing the Higgs branches of the effective theory (4.3) by equations for its gauge-invariant fields and then finding a change of variables mapping them to the meson and baryon constraints (2.19), (2.20), (2.23), and (2.24) derived in section 2.3.4.

The D-term vacuum equations (4.6) are solved by writing all the holomorphic invariants of the complexified gauge group. An overcomplete basis of such invariant fields is

$$N_i^j \equiv q^j \widetilde{q}_i,$$

$$E_k \equiv e_k \widetilde{e}_k,$$

$$b^{i_1 \ldots i_{\widetilde{n}_c}} \equiv q_{a_1}^{i_1} \cdots q_{a_{\widetilde{n}_c}}^{i_{\widetilde{n}_c}} \epsilon^{a_1 \ldots a_{\widetilde{n}_c}} \prod_{k=1}^{n_c - \widetilde{n}_c} e_k, \tag{4.8}$$

$$\widetilde{b}_{i_1 \ldots i_{\widetilde{n}_c}} \equiv \widetilde{q}_{i_1}^{a_1} \cdots \widetilde{q}_{i_{\widetilde{n}_c}}^{a_{\widetilde{n}_c}} \epsilon_{a_1 \ldots a_{\widetilde{n}_c}} \prod_{k=1}^{n_c - \widetilde{n}_c} \widetilde{e}_k.$$

Note that gauge invariance of these requires the $U(1)$ charges of q to be those of (4.3). The first two F-term equations in (4.7) imply

$$E_k = \frac{1}{\widetilde{n}_c} \text{Tr} N. \tag{4.9}$$

The constraints satisfied by N, b, and $\widetilde{b}$, by virtue of their definitions (4.8) are of the same nature as those satisfied by the M, B, and $\widetilde{B}$ fields discussed in section 2.3.4. By similar arguments their D-term constraints can be reduced to

$$(*b)\widetilde{b} = \left(\frac{1}{\widetilde{n}_c} \text{Tr} N \right)^{n_c - \widetilde{n}_c} * (N^{\widetilde{n}_c}), \tag{4.10}$$

$$N \cdot *b = N \cdot *\widetilde{b} = 0, \tag{4.11}$$

when neither b nor $\widetilde{b}$ (nor therefore $\mathrm{Tr}N$) vanishes. In this case note that (4.10) and (4.11) imply

$$\mathrm{rank}(N) \leq \widetilde{n}_c. \tag{4.12}$$

When both $b = \widetilde{b} = 0$, all the D-term constraints except (4.10) and (4.12) become trivial. Note that, unlike the case for the microscopic constraints found in section 2.3.4, (4.10) does *not* imply (4.12) when $b = \widetilde{b} = 0$, and thus (4.12) must be added to the list of independent constraints (though it is only independent for the non-baryonic branches).[11]

The remaining F-term equations in (4.7) imply

$$N' \cdot b = N' \cdot \widetilde{b} = 0, \tag{4.13}$$

$$N \cdot N' = 0, \tag{4.14}$$

as well as another constraint which is just a contraction of (4.10). We have defined

$$(N')^i_j \equiv N^i_j - \frac{1}{\widetilde{n}_c}(\mathrm{Tr}N)\delta^i_j. \tag{4.15}$$

Thus, (4.10)–(4.14) form a complete set of constraints for the Higgs branches of the IR–effective theory (4.3).

We can map the constraints for the microscopic theory into the constraints of the IR–effective theory by the change of variables

$$M = N', \qquad B = (-)^{n_c} * \widetilde{b}, \qquad \widetilde{B} = (-)^{n_c} * b, \tag{4.16}$$

whose inverse is

$$N = M', \qquad b = (-)^{\widetilde{n}_c} * \widetilde{B}, \qquad \widetilde{b} = (-)^{\widetilde{n}_c} * B. \tag{4.17}$$

Note that the mapping (4.16) implies the normalization of the baryon number given in (4.3). To prove the claim that the branches of the IR–effective theory (4.3) are a subset of

---

[11] The case when only one of b and $\widetilde{b}$ vanishes requires a separate argument. As noted in section 2.3.4, there are more constraints for this case; however, their solutions are just submanifolds of the baryonic branch, as can be seen from explicit solutions of (4.6) and (4.7) for the squark vevs.

the branches of the original microscopic $SU(n_c)$ theory, we must show that (4.10)–(4.14) imply (2.19), (2.20), (2.23), and (2.24).

It follows immediately from (4.16) and (4.17) that (4.14) is equivalent to (2.24), (4.11) to (2.23), and (4.13) to (2.20). Thus, we only need to show that (2.19) is satisfied. On a branch with both b and $\widetilde{b}$ non-zero, we must have, by (4.10), $\mathrm{Tr}N \neq 0$, implying $\mathrm{Tr}M \neq 0$. From section 2.3.4, the solution to (2.24) with $\mathrm{Tr}M \neq 0$ is

$$
M = \begin{pmatrix} \rho & & & & & \lambda_1 & & \\ & \ddots & & & & & \ddots & \\ & & \ddots & & & & & \\ & & & \rho & 0 & \cdots & & \lambda_{\widetilde{n}_c} \\ & & & & & & & 0 \\ & & & & & & & \\ & & & & & & & \end{pmatrix} \tag{4.18}
$$

where the ρ block is $n_c \times n_c$, implying

$$
N = \begin{pmatrix} & & & \lambda_1 & & & \\ & & & & \ddots & & \\ & & & & & & \lambda_{\widetilde{n}_c} \\ & & 0 & \cdots & & 0 \\ & & -\rho & & & \\ & & & & \ddots & \\ & & & & & -\rho \end{pmatrix} . \tag{4.19}
$$

But it is now straight-forward to check using these solutions that $(*B)\,\widetilde{B} = b\,(*\widetilde{b}) = (-\rho)^{n_c - \widetilde{n}_c} * (N^{\widetilde{n}_c}) = *(M^{n_c})$, showing that (2.19) is satisfied. When both b and $\widetilde{b}$ vanish, either $\mathrm{Tr}N = 0$ or $\mathrm{rank}(N) < \widetilde{n}_c$. In either case (4.14) then implies $\mathrm{rank}(M) < n_c$, so that (2.19) is again satisfied.

So, we have shown that the Higgs branches of the IR–effective theory (4.3) are a subset of those of the microscopic theory. Furthermore, we have seen explicitly that the baryonic branch is a solution to both sets of constraints. The difference between the two sets is in the non-baryonic branches. In particular, on the non-baryonic branches (since $\mathrm{Tr}N = 0$) we have from (4.14) that

$$
N = M = \begin{pmatrix} & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_r \\ & & & \\ & & & \end{pmatrix}, \qquad r \leq [n_f/2]. \tag{4.20}
$$

However, the "extra" constraint (4.12) implies that of the branches (4.20) only those with $r \leq \widetilde{n}_c$ survive. Thus the IR–effective theory at the root of the baryonic branch includes a subset of the non-baryonic branches of the microscopic theory.

4.3. N=2 Duality and Flowing Down in Flavors

In the previous subsection we determined the IR–effective theory at the root of the baryonic branch using the unbroken $\mathbb{Z}_{n_c - \widetilde{n}_c}$ symmetry to argue that there should be an unbroken $SU(\widetilde{n}_c) \times U(1)^{n_c - \widetilde{n}_c}$ gauge group, and demanding singlets charged under the $U(1)$'s so the Higgs branch would exist, leading to the quantum numbers (4.3). We then checked that (4.3) is indeed the effective theory at the baryonic root by showing that its Higgs branches are the baryonic Higgs branch as well as $\widetilde{n}_c$ of the $[n_f/2]$ non-baryonic branches.

In this subsection we will perform a further check which illuminates the connection between the effective theory at the baryonic root and a conjectured S–duality of a finite $N=2$ theory. In particular, we show that (4.3) can be obtained by starting with the scale-invariant theory with $2n_c$ flavors and flowing down to n_f flavors by giving bare masses to $n_c - \widetilde{n}_c$ quarks. We determine the form of this bare mass term in the IR–effective theory using the conjectured S-duality of the finite $N=2$ theory.

Consider the scale-invariant theory with $2n_c$ flavors, coupling τ, and bare masses m_i. The exact solution for this theory [15] shows it has a symmetry implying an equivalent description in terms of a dual theory with coupling and masses replaced by

$$\tau \to \widetilde{\tau} = -1/\tau,$$
$$m_i \to \widetilde{m}_i = m_i - 2m_S, \quad (4.21)$$

where m_S is the flavor-scalar mass defined by

$$m_S \equiv \frac{1}{n_f} \sum_{i=1}^{n_f} m_i. \quad (4.22)$$

Note that under the duality transformation $\widetilde{m}_S = -m_S$ while the adjoint mass $\widetilde{m}_{Ai} = \widetilde{m}_i - \widetilde{m}_S = m_i - m_S = m_{Ai}$ is invariant. This fact has a natural interpretation in terms of the non-renormalization theorem of section 3. Since we think of m_i as background vector

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superfields, their couplings are constrained by gauge invariance and the coefficients in the superpotential (2.2) is the corresponding charge. For m_S that charge is the baryon number. Therefore, the transformation (4.21) means that the baryon number of the magnetic quarks is opposite to that of the electric quarks.

We flow down to the theory with $n_f < 2n_c$ by turning on $n_c - \widetilde{n}_c$ quark masses of order M:

$$m_i = (\underbrace{Mx_i}_{n_c - \widetilde{n}_c} \; ; \; \underbrace{0}_{n_f}). \tag{4.23}$$

The resulting theory should not depend on the way we decouple the $n_c - \widetilde{n}_c$ flavors, so the x_i should be taken to be arbitrary numbers satisfying $|x_i| \gtrsim 1$. The massive quarks decouple in the limit $q \equiv e^{i\pi\tau} \to 0$ and $M \to \infty$ keeping $\Lambda \equiv q^{1/(n_c - \widetilde{n}_c)} M$ fixed. The dual of this theory is at very strong coupling with masses $\widetilde{m}_i \sim M$:

$$\widetilde{m}_i = \Big(\underbrace{Mx_i - \frac{M}{n_c} \sum x}_{n_c - \widetilde{n}_c} \; ; \; \underbrace{-\frac{M}{n_c} \sum x}_{n_f} \Big). \tag{4.24}$$

Our strategy for determining the spectrum of the effective theory at the root of the baryonic branch will be to study this dual theory at weak coupling instead of strong coupling. The point is that since this theory is IR–free, it will exist along whole submanifolds in parameter space $(\widetilde{m}_i, \widetilde{\tau})$ which can plausibly be followed out to weak coupling.

A unique point ϕ on the Coulomb branch of the dual theory is determined by the requirement of IR–freedom and the existence of a purely hypermultiplet Higgs branch. Since for generic x_i the first $n_c - \widetilde{n}_c$ masses in (4.24) are all different, by tuning ϕ each can contribute at most one massless flavor which, to be IR–free, must be charged only under $U(1)$ factors. In order to have a purely hypermultiplet Higgs branch, however, we must have at least as many massless singlets as $U(1)$ factors. These conditions determine the diagonal ϕ vev

$$\phi = \Big(\underbrace{-Mx_i + \frac{M}{n_c} \sum x}_{n_c - \widetilde{n}_c} \; ; \; \underbrace{\frac{M}{n_c} \sum x}_{\widetilde{n}_c} \Big), \tag{4.25}$$

giving an $SU(\widetilde{n}_c) \times U(1)^{n_c - \widetilde{n}_c}$ effective theory with n_f massless squarks, and $n_c - \widetilde{n}_c$ massless singlets each of which is charged under a single $U(1)$ factor. Normalizing these $U(1)$'s

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so that each singlet has charge -1, the squarks must then have charge $1/\widetilde{n}_c$ under each of the $U(1)$ factors, as follows from the tracelessness of each factor inherited from its embedding in the original $SU(n_c)$ group. We thus recover the baryonic root effective theory (4.3).

We can check the baryon number of the dual squarks in (4.3) as well, using their coupling to the flavor-scalar mass. Consider adding in additional small bare masses m_i' for the squarks in the microscopic theory. The bare masses m_i, their duals $\widetilde{m}_i$, and the vev for ϕ are then

$$
m_i = (\ \underbrace{Mx_i}_{n_c - \widetilde{n}_c} \ ;\ \underbrace{m_{\hat{i}}'}_{n_f}\ ),
$$

$$
\widetilde{m}_i = \Bigg(\ \underbrace{Mx_i - \frac{M}{n_c}\sum x - \frac{1}{n_c}\sum m'}_{n_c - \widetilde{n}_c}\ ;\ \underbrace{m_{\hat{i}}' - \frac{M}{n_c}\sum x - \frac{1}{n_c}\sum m'}_{n_f}\ \Bigg),
$$

(4.26)

$$
\phi = \Bigg(\ \underbrace{-Mx_i + \frac{M}{n_c}\sum x + \frac{1}{n_c}\sum m'}_{n_c - \widetilde{n}_c}\ ;\ \underbrace{\frac{M}{n_c}\sum x - \frac{n_c - \widetilde{n}_c}{n_c\widetilde{n}_c}\sum m'}_{\widetilde{n}_c}\ \Bigg),
$$

with $|M| \gg |m_i'|$. On the Coulomb branch of the dual theory near ϕ the effective theory is $SU(\widetilde{n}_c) \times U(1)^{n_c - \widetilde{n}_c}$ with n_f squarks with masses

$$
\widetilde{m}_i' = m_i' - \frac{1}{\widetilde{n}_c}\sum m',
$$

(4.27)

and we tuned ϕ so the $n_c - \widetilde{n}_c$ singlets are exactly massless. (The fact that we can shift these masses away shows that they are not true parameters describing the vacuum in question.) The expression for the quark masses in the dual theory (4.27) implies the duality transformation on the flavor-scalar mass

$$
m_S \to \widetilde{m}_S = -\frac{n_c}{\widetilde{n}_c} m_S.
$$

(4.28)

Recalling our interpretation of the masses as the vevs of vector multiplets gauging the flavor symmetries, it follows that the coefficient of the scalar mass in the superpotential should be identified with the baryon number of the squarks. In the microscopic theory the singlet mass couples as

$$
\mathcal{W}_{\mathrm{micro}}/\sqrt{2} = +m_S\, \mathrm{tr} Q{\cdot}\widetilde{Q},
$$

(4.29)

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normalizing the baryon number of the squarks to 1. In the dual theory, we have

$$\mathcal{W}_{\text{dual}}/\sqrt{2} = +\widetilde{m}_S \, \text{tr} q \cdot \widetilde{q} = -\frac{n_c}{\widetilde{n}_c} m_S \, \text{tr} q \cdot \widetilde{q}, \qquad (4.30)$$

implying the dual squarks have baryon number $-n_c/\widetilde{n}_c$.

We could also have checked the assignment of charges (4.3) by flowing down one flavor at a time using (4.27). It is worth pointing out that one finds in this way the spectrum for the extreme $n_f = n_c$ case (which is not covered in (4.3)): [12]

$$
\begin{array}{cccccc}
 & U(1)_1 & \times & \cdots & \times & U(1)_{n_c-1} & \times & U(1)_B \\
e_0 & 1 & & \cdots & & 1 & & -n_c \\
e_1 & -1 & & \cdots & & 0 & & 0 \\
\vdots & \vdots & & \ddots & & \vdots & & \vdots \\
e_{n_c-1} & 0 & & \cdots & & -1 & & 0
\end{array}
\qquad (4.31)
$$

5. Quantum Coulomb Branch

Though the arguments of the last section determined the low energy physics on the Coulomb branch at the roots of the Higgs branches, they did not determine the positions of these roots on the Coulomb branch. In this section we will locate these points using the exact solution for the effective theory on the Coulomb branch. While, on the one hand, locating these points can be viewed as a check on the arguments of the last section, on the other hand, given that those arguments only required semi-classical physics, this section may be better viewed as a check on the exact solutions.

5.1. Review of the Exact Solution

The generic vacuum on the Coulomb branch is a $U(1)^{n_c-1}$ pure Abelian gauge theory characterized by an effective coupling τ_{ij} between the ith and jth $U(1)$ factors. Due to the ambiguity of electric-magnetic duality rotations in each $U(1)$ factor, as well as changes of basis among the $U(1)$'s, the τ_{ij} form a section of an $Sp(n_c-1, \mathbb{Z})$ bundle over the Coulomb branch [6].

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[12] It may be interesting to note that these quantum numbers (for $n_c = 16$) correspond to the massless spectrum at the conifold point studied in [23]. Is there a non-Abelian (*e.g.* a heterotic dual) theory whose long-distance behavior leads to this spectrum at that point?

Locally the Coulomb branch is n_c-1 complex dimensional, corresponding to the vevs of the n_c-1 $U(1)$ vector multiplets. Globally, it is found in [13-16] that the Coulomb branch can be characterized by n_c complex numbers $\Phi = (\phi_1, \ldots, \phi_{n_c})$ (up to permutations) whose sum vanishes. At weak coupling these coordinates are identified with the coordinates of the adjoint scalar vev in the Cartan subalgebra of $SU(n_c)$.

An explicit description of the Coulomb branch was found in [13-16] by associating to each point of the Coulomb branch a genus n_c-1 Riemann surface whose complex structure is the low energy coupling τ_{ij}. This family of Riemann surfaces is described by complex curves $y^2 = P_{2n_c}(x)$ where P_{2n_c} is a polynomial of degree $2n_c$ in x whose coefficients are polynomials in the moduli ϕ_a and the parameters—the bare quark masses m_i, and the strong-coupling scale Λ or bare coupling τ (in the latter case the dependence is through θ functions). Such a double cover of the complex x-plane branched over $2n_c$ points describes a genus n_c-1 hyperelliptic Riemann surface. When two or more of the branch points (the zeros of P_{2n_c}) collide as we vary the moduli or parameters, the Riemann surface degenerates. Such a singularity in the effective action corresponds to additional $N{=}2$ multiplets becoming massless.

The mass of any BPS saturated state in the theory is given by the period of a certain one-form around a cycle on the Riemann surface [6]. The homology class of the cycle encodes the $U(1)$ charges of the state. States corresponding to non-intersecting cycles are "mutually local" in the sense that they can be taken to be simultaneously solely electrically charged under the $U(1)$ factors. Thus, when only two branch points collide the cycle encircling the two points vanishes, corresponding to vanishing mass for hypermultiplet states with $U(1)$ charges proportional to the homology class of the vanishing cycle. Near such a point, the curve will have the form $y^2 = P_{2n_c-2}(x) \cdot (x-x_1)^2$ for some P_{2n_c-2} and x_1. The number of hypermultiplets (weighted by their charges) is determined by the monodromy of the periods as we move ϕ about the singular point. In the generic case in which there is precisely one massless hypermultiplet, we will refer to such a curve as having a single hypermultiplet singularity. Similarly, if n_s independent pairs of zeros of P_{2n_c} collide, the curve is $y^2 = P_{2n_c-2n_s} \cdot \prod_{a=1}^{n_s}(x-x_a)^2$, and has generically an n_s (mutually local) hypermultiplet singularity.

When more than two zeros collide at a point, more complicated sets of states become massless. In some cases the resulting effective theories are non-trivial fixed points [11,12]. Others may be IR–free or scale-invariant non-Abelian Coulomb points. We will now develop a "dictionary" for identifying the latter type of singularity from the form of the curve. (The beginnings of such a dictionary for the non-trivial fixed points appears in [12].)

The curve for the $n_f = 2n_c$ scale-invariant theory is [15]

$$y^2 = \prod_{a=1}^{n_c} (x - \phi_a)^2 + h(h+2) \prod_{i=1}^{n_f} (x + hm_S + m_i), \qquad n_f = 2n_c \qquad (5.1)$$

where $m_S = (1/n_f) \sum m_i$ is the singlet mass, and $h(q) = 32q + \mathcal{O}(q^2)$ where $q = e^{i\pi\tau}$, is a specific modular function of τ: $h(\tau) = 2\theta_1^4/(\theta_2^4 - \theta_1^4)$. (We define the $\theta_i(\tau)$ as in [5].) The duality symmetry of the curve under the transformation (4.21) described earlier follows from the modular transformation property $h \to -(h+2)$ under $\tau \to -1/\tau$ [15].

One can deduce the $n_f > 2n_c$ curves by starting with the scale-invariant theory with $n_f/2$ colors and breaking the gauge group at a scale M down to $SU(n_c)$ on the Coulomb branch at weak coupling ($h \sim 0$) with $\Lambda^{2n_c - n_f} = 16qM^{2n_c - n_f}$ fixed. (The factor of 16 is for later convenience.) This corresponds to setting $|\phi_a| \ll |\phi_k| \sim M$ for $a = 1, \ldots, n_c$ and $k = n_c + 1, \ldots, n_f/2$ with $\sum_a \phi_a = 0$. Such a procedure makes sense since the $SU(n_c)$ theory with n_f flavors is IR–free, so for any Λ the region of the Coulomb branch in question is weakly coupled. In the limit $q \to 0$ we find the curve

$$y^2 = \prod_{a=1}^{n_c} (x - \phi_a)^2 + 4\Lambda^{2n_c - n_f} \prod_{i=1}^{n_f} (x + m_i), \qquad n_f > 2n_c, \qquad (5.2)$$

for $|x|, |\phi_a|, |m_i| \ll M$. M is the scale at which the description of the physics by the curve (5.2) must break down (it becomes sensitive to the physics that regularizes it in the UV at scales above M). Λ is the scale where this IR–free theory becomes strongly coupled. Note that at weak coupling $M \ll \Lambda$, and the cutoff scale M cannot be determined from the curve (5.2) alone. These considerations are reflected in the fact that although (5.2) is supposed to describe the Coulomb branch of an $SU(n_c)$ theory, it involves a polynomial in x of degree $n_f > 2n_c$. But the $2n_c - n_f$ "extra" zeros of the right-hand side of (5.2) are at $x \sim \Lambda$, and so are of no consequence in the region of validity $|x| \ll M \ll \Lambda$ of (5.2).

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The curves for the asymptotically-free theories with $n_f < 2n_c$ are obtained from the curve for the scale-invariant theory by integrating out some of the flavors. Thus, we can determine the curve for $n_f = 2n_c-1$ by taking $q \to 0$ and one of the masses, say m_{2n_c}, large such that $\Lambda = 16qm_{2n_c}$ stays fixed, thus decoupling that flavor. Λ becomes the strong-coupling scale of the resulting AF theory. The curve becomes for $|x| \ll |m_{2n_c}|$

$$y^2 = \prod_{a=1}^{n_c}(x-\phi_a)^2 + 4\Lambda \prod_{i=1}^{n_f}\left(x+m_i+\frac{\Lambda}{n_c}\right), \qquad n_f = 2n_c - 1, \qquad (5.3)$$

thus describing the Coulomb branch of the $n_f = 2n_c-1$ theory for all values of x, ϕ_a, and m_i. For two or more large masses, say $m_i \to \infty$ for $n_f+1 \le i \le 2n_c$, take $q \to 0$ such that $\Lambda^{2n_c-n_f} = 16q\prod_{i=n_f+1}^{2n_c} m_i$ stays fixed. The curve becomes for $|x| \ll \min_{n_f+1 \le i \le 2n_c}\{|m_i|\}$

$$y^2 = \prod_{a=1}^{n_c}(x-\phi_a)^2 + 4\Lambda^{2n_c-n_f}\prod_{i=1}^{n_f}(x+m_i), \qquad n_f \le 2n_c - 2, \qquad (5.4)$$

describing the Coulomb branch of the $n_f \le 2n_c-2$ theories.

5.2. The Non-Baryonic Roots

From the discussion of section 4.1, the non-baryonic roots have unbroken $SU(r) \times U(1)^{n_c-r}$ gauge symmetry for $r \le [n_f/2]$, with n_f massless quarks. This suggests we look at a submanifold of the Coulomb branch of the form

$$\Phi_{\mathrm{nbb}} = (0,\ldots,0,\varphi_1,\ldots,\varphi_{n_c-r}), \qquad \sum \varphi_a = 0, \qquad |\varphi_a| \sim M. \qquad (5.5)$$

The curve with zero masses for $n_f \le 2n_c-2$ is $y^2 = \prod_a^{n_c}(x-\phi_a)^2 + 4\Lambda^{2n_c-n_f}x^{n_f}$. For $\Phi = \Phi_{\mathrm{nbb}}+\delta\Phi$ near Φ_{nbb}, $\delta\Phi = (\phi_1,\ldots,\phi_r,0,\ldots,0)$, $\sum\phi_a = 0$, and $|\phi_a|, |x| \ll M$, the curve becomes approximately

$$y^2 = \prod_a^{r}(x-\phi_a)^2 M^{2(n_c-r)} - 4\Lambda^{2n_c-n_f}x^{n_f}, \qquad (5.6)$$

which we recognize as the curve for $SU(r)$ with n_f flavors, confirming (5.5) as the coordinates of the non-baryonic roots. (For $n_f = 2n_c-1$ the coordinates have to be shifted to $\Phi_{\mathrm{nbb}} = (0,\ldots,0,\varphi_1-\frac{1}{r}\Lambda,\ldots,\varphi_{n_c-r}-\frac{1}{r}\Lambda) - \frac{1}{n_c}\Lambda(1,\ldots,1)$. Then, for $\widetilde{x} \equiv x+\frac{1}{n_c}\Lambda \ll \Lambda$, (5.3) becomes approximately (5.6).)

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Now we set $\delta\Phi = 0$ and look for points on the Φ_{nbb} submanifold for which we get the maximal number of mutually local massless monopoles. The curve becomes

$$y^2 = x^{2r} \left\{ \prod_a^{n_c - r} (x - \varphi_a)^2 + 4\Lambda^{2n_c - n_f} x^{n_f - 2r} \right\}. \tag{5.7}$$

Since we have $n_c - r - 1$ parameters φ_a at our disposal, we can (generically) tune $2(n_c - r - 1)$ zeros of (5.7) to double up. By choosing a basis of contours on the x-plane with $n_c - r - 1$ which each loop around one of the doubled zeroes and the rest which loop in various ways around the $2r$ zeros corresponding to the $SU(r)$ factor, we find (generically) the non-baryonic root charges given in (4.1). It is difficult to determine the actual values of φ_a for which this coincidence of zeros is realized. From the discrete $\mathbb{Z}_{2n_c - n_f}$ symmetry of the curve (5.7), it follows that there will generically be $2n_c - n_f$ such points. When $r = n_f/2$, the curve in braces in (5.7) describes the pure $SU(n_c - n_f/2)$ Yang-Mills theory, and explicit coordinates for its multimonopole points have been computed in [24]. When $r \le n_f - n_c$, there is in fact a point at which the discrete symmetry is unbroken and there are $n_c - r$ massless hypermultiplets. This point is the baryonic root found in the next subsection. For $r < n_f - n_c$ there are also points as described above with $n_c - r - 1$ massless hypermultiplets and a broken discrete symmetry. Thus we see (as discussed following (4.20)) that quantum mechanically the nonbaryonic branches with $r \le n_f - n_c$ intersect the baryonic branch.

5.3. The Baryonic Root

From the discussion in section 4.1, we expect the effective theory at the baryonic root to have an $SU(n_f - n_c) \times U(1)^{n_c - \widetilde{n}_c}$ gauge symmetry with n_f massless quarks, and to be invariant under the $\mathbb{Z}_{n_c - \widetilde{n}_c}$ discrete symmetry on the Coulomb branch. Thus (for $n_f \le 2n_c - 2$) we take

$$\Phi_{\text{bb}} = (0, \ldots 0, \varphi\omega, \ldots, \varphi\omega^{n_c - \widetilde{n}_c}), \tag{5.8}$$

where $\omega = \exp\{2\pi i/(n_c - \widetilde{n}_c)\}$. Indeed, for $\Phi = \Phi_{\text{bb}} + \delta\Phi$ near Φ_{nbb}, $\delta\Phi = (\phi_1, \ldots, \phi_{\widetilde{n}_c}, 0, \ldots, 0)$, $\sum \phi_a = 0$, and $|\phi_a|, |x| \ll \varphi$, the curve becomes approximately

$$y^2 = \prod_a^{\widetilde{n}_c} (x - \phi_a)^2 \varphi^{2(n_c - \widetilde{n}_c)} + 4\Lambda^{n_c - \widetilde{n}_c} x^{n_f}, \tag{5.9}$$

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which we recognize as the curve for $SU(\widetilde{n}_c)$ with n_f massless flavors.

We now determine φ such that $n_c-\widetilde{n}_c$ mutually local hypermultiplets become massless. Setting $\delta\Phi = 0$, the curve becomes

$$y^2 = x^{2\widetilde{n}_c}\left[(x^{n_c-\widetilde{n}_c} - \varphi^{n_c-\widetilde{n}_c})^2 + 4\Lambda^{n_c-\widetilde{n}_c}x^{n_c-\widetilde{n}_c}\right].\qquad(5.10)$$

At the point

$$\varphi = \Lambda \qquad (5.11)$$

the right hand side of (5.10) becomes a perfect square, giving the right singularity structure to describe $n_c-\widetilde{n}_c$ massless hypermultiplets. This is thus the point we want.[13] It is easy to check that there is indeed precisely one hypermultiplet of each set of charges, by computing the monodromy. (For $n_f = 2n_c-1$ we have to shift $\Phi_{\rm bb}$ to $\Phi_{\rm bb} = (0,\ldots,0,n_c\Lambda) - \Lambda(1,\ldots,1)$.)

Furthermore, if we take a basis of cycles on the x-plane with $n_c-\widetilde{n}_c$ cycles α_i which each encircle one of the $n_c-\widetilde{n}_c$ pairs of zeros of (5.10), and the rest of the cycles looping in various ways around the $2\widetilde{n}_c$ zeros associated with the $SU(\widetilde{n}_c)$ factor, we see that the singlets are each charged under only one $U(1)$ factor. For small ϕ_a in (5.9) the $2\widetilde{n}_c$ degenerate zeros split. Consider a set of $\widetilde{n}_c$ non-intersecting contours γ_a each looping once around one pair of these slightly split zeros. Thus each γ_a corresponds to a light quark. A contour deformation on the x-plane shows that the set $\{\alpha_i, \gamma_a\}$ of contours satisfy one relation $\sum_i \alpha_i + \sum_a \gamma_a = 0$. Since the γ_a do not sum to zero, they do not correspond to a basis of $U(1)$'s in the Cartan subalgebra of the $SU(\widetilde{n}_c)$ factor. We can define an alternative basis of cycles $\{\alpha_i, \beta_a\}$ with $\beta_a = \gamma_a + \frac{1}{\widetilde{n}_c}\sum_i \alpha_i$ which does have this property. In this basis each quark contour γ_a has a $-\frac{1}{\widetilde{n}_c}\alpha_i$ piece, and so we conclude that the quarks are each charged under all the $U(1)$ factors with $-1/\widetilde{n}_c$ of the corresponding singlet charge. Thus we have reproduced the quantum numbers for the baryonic root derived by other arguments in section 4.

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[13] We can also check that there are no such multimonopole points on the larger $\mathbb{Z}_{n_c-\widetilde{n}_c}$ invariant submanifolds for $n_f \geq \frac{3}{2}n_c$ mentioned in the footnote after Eq. (4.2). On this submanifold the curve becomes $y^2 = x^{2n_c-2k\nu} \cdot (\prod_{i=1}^{k}(x^\nu - \varphi_i^\nu)^2 + 4\Lambda^\nu x^{(2k-1)\nu})$, where $\nu \equiv n_c - \widetilde{n}_c$. It is easy to show that the right hand side becomes a perfect square only for all but one $\varphi_i = 0$, which restricts us back to the submanifold (5.8).

6. Breaking to $N=1$ Supersymmetry

In this section we break to $N=1$ supersymmetry by turning on a bare mass μ for the adjoint superfield Φ. In the microscopic theory in the limit $\mu \to \infty$ this leads to $N=1$ $SU(n_c)$ super–QCD. By performing this breaking in the macroscopic effective theories at the roots of the Higgs branches we find instead $SU(\tilde{n}_c)$ super–QCD with some extra gauge singlets. This is the dual formulation of $N=1$ QCD suggested in [7].

6.1. Breaking in the Microscopic Theory

Since Φ is part of the $N=2$ vector multiplet, giving it a mass explicitly breaks $N=2$ supersymmetry. In the microscopic theory, this corresponds to an $N=1$ theory with a superpotential

$$\mathcal{W} = \sqrt{2}\,\mathrm{tr}(Q\Phi\tilde{Q}) + \frac{\mu}{2}\mathrm{tr}(\Phi^2). \tag{6.1}$$

For $\mu \gg \Lambda$ we can integrate Φ out in a weak-coupling approximation, obtaining an effective quartic superpotential

$$\mathcal{W}_{\prime} = -\frac{1}{\mu}\left(\mathrm{tr}(Q\tilde{Q}Q\tilde{Q}) - \frac{1}{n_c}\mathrm{tr}(Q\tilde{Q})\mathrm{tr}(Q\tilde{Q})\right). \tag{6.2}$$

In the limit $\mu \to \infty$ this superpotential becomes negligible, and we find $N=1$ $SU(n_c)$ super–QCD with n_f flavors and no superpotential.

Note that if the strong coupling scale of the $N=2$ theory is Λ, then by a one-loop matching, the scale of the $N=1$ theory will be $\Lambda_1^{3n_c-n_f} = \mu^{n_c}\Lambda^{2n_c-n_f}$. The appropriate scaling limit will send $\mu \to \infty$ and $\Lambda \to 0$ keeping Λ_1 fixed. The model is described by the $N=1$ super–QCD theory on scales between μ and Λ_1, below which the strongly-coupled dynamics of the $N=1$ theory takes over.

6.2. Breaking in the Low-Energy Effective Theories

We can also study the breaking to $N=1$ by beginning with $\mu \ll \Lambda$. In this case we should study the low-energy $N=2$ theory obtained in the previous sections and the effects of μ in this theory. $N=1$ supersymmetry prevents a phase transition as we vary μ, hence we should obtain the same result as that obtained for $\mu \gg \Lambda$ in the previous subsection (see Fig. 3).

It is easy to see that generic vacua of the $N=2$ theory are lifted by nonzero μ; we will show that the baryonic root, as well as the special points we have found on the nonbaryonic roots, are not. There may be other points that survive the breaking, but the light fields at these points will not include non-Abelian gauge multiplets. We thus study the effects of the breaking to $N=1$ in the effective theories at the roots of the Higgs branches. We saw in section 4 that these effective theories have unbroken gauge groups of the form $SU(r) \times U(1)^{n_c-r}$. Let ϕ denote the adjoint scalar in the $SU(r)$ factor, and ψ_k the adjoint scalars for each of the $U(1)$ factors. Then a microscopic mass term $(\mu/2)\mathrm{tr}\Phi^2$ becomes $\mu(\Lambda \sum_i x_i \psi_i + \frac{1}{2}\mathrm{tr}\phi^2 + \ldots)$, where the dots denote higher-order terms, and x_i are dimensionless numbers. From the Φ vevs breaking $SU(n_c) \to SU(r) \times U(1)^{n_c-r}$ found in section 5, we see that all $x_i \sim 1$.

Now let us examine what happens to the effective theory at the roots of the non-baryonic branches when we turn on such a mass term. The first thing to note is that at any point on the non-baryonic root for which there are fewer than n_c-r-1 massless singlets, e_k, charged under the $U(1)$'s, then the $N=2$ vacuum is lifted. This can be seen as follows. If there are n_s singlets with $n_s < n_c-r-1$, a basis of the $U(1)$'s can be chosen to diagonalize the charges of the singlets and the quarks, and the superpotential becomes

$$\mathcal{W}_{\mathrm{nbb}} = \sqrt{2}\,\mathrm{tr}(q\phi\widetilde{q}) + \sqrt{2}\,\psi_0\mathrm{tr}(q\widetilde{q}) + \sqrt{2}\sum_{k=1}^{n_s}\psi_k e_k\widetilde{e}_k + \mu\Big(\Lambda \sum_{i=0}^{n_c-r-1} x_i\psi_i + \frac{1}{2}\mathrm{tr}\phi^2\Big). \quad (6.3)$$

The F-term equations following from taking derivatives with respect to the ψ_i then have no solution.

Therefore only the special vacua (4.1) on the non-baryonic roots with $n_s = n_c-r-1$ lead to $N=1$ vacua. Then the ψ_i F-term equations imply $e_k\widetilde{e}_k \neq 0$, while the e_k equations imply that all $\psi_i = 0$ except ψ_0. Thus when $\mu \neq 0$ these fields can be integrated-out, leaving the effective superpotential

$$\mathcal{W}'_{\mathrm{nbb}} = \sqrt{2}\,\mathrm{tr}(q\phi\widetilde{q}) + \frac{\mu}{2}\mathrm{tr}\phi^2 + \psi_0\left(\sqrt{2}\,\mathrm{tr}(q\widetilde{q}) + \mu\Lambda\right), \quad (6.4)$$

for an $N=1$ $SU(r) \times U(1)$ super–QCD, with $r \leq [n_f/2]$. The q and ϕ F-term equations then imply a trade-off between the rank of the unbroken gauge group and the number of

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massless singlets. The general solutions up to color and flavor rotations are as follows.

$$\psi_0 = \frac{\ell-r}{\ell r}\Lambda, \qquad \ell \in \{0, 1, \ldots, r\},$$

$$\phi = \frac{\Lambda}{\ell r}\text{diag}(\underbrace{r-\ell, \ldots, r-\ell}_{\ell}, \underbrace{-\ell, \ldots, -\ell}_{r-\ell}),$$

$$q = \begin{pmatrix} \kappa_1 & & & \\ & \ddots & & \\ & & \kappa_\ell & \end{pmatrix}, \qquad \kappa_a \in \mathbb{R}^+, \qquad (6.5)$$

$${}^t\widetilde{q} = \begin{pmatrix} \widetilde{\kappa}_1 & & & \lambda_1 & & \\ & \ddots & & & \ddots & \\ & & \widetilde{\kappa}_\ell & & & \lambda_\ell \end{pmatrix}, \qquad \lambda_a \in \mathbb{R}^+,$$

where $\kappa_a\widetilde{\kappa}_a = -\mu\Lambda/\ell$, independent of a, and $\lambda_a^2 = \kappa_a^2 - \widetilde{\kappa}_a^2$. This classical moduli space of solutions all have unbroken $SU(r-\ell)$ gauge symmetry with $\ell(n_f-\ell)$ massless singlets, and no light charged matter. The corresponding quantum theories have no IR gauge group, since $N=1$ Yang-Mills theory is known to be confining. The massless chiral multiplets along these flat directions—the meson fields $N_j^i = q^i\widetilde{q}_j$—should be identified with the meson fields M in the dual $N=1$ theory of [7], since they are gauge singlets, have zero baryon number, and transform in the adjoint plus singlet representations of $SU(n_f)$ flavor. Furthermore, since $\text{rank}(N) \leq [n_f/2]$, N cannot be identified with the meson field $\tilde{M}_j^i$ bilinear in the dual $N=1$ quarks since $\text{rank}(\tilde{M}) \leq \widetilde{n}_c < [n_f/2]$. Therefore N must be (part of) the "extra" singlet degrees of freedom M found in the $N=1$ dual of [7].

Now let us examine the breaking of the effective theory at the baryonic root. In this case, from (4.3), the superpotential is

$$\mathcal{W}_{\text{bb}} = \sqrt{2}\,\text{tr}(q\phi\widetilde{q}) + \frac{\sqrt{2}}{\widetilde{n}_c}\text{tr}(q\widetilde{q})\Big(\sum_{k=1}^{n_c-\widetilde{n}_c}\psi_k\Big) - \sqrt{2}\sum_{k=1}^{n_c-\widetilde{n}_c}\psi_k e_k\widetilde{e}_k + \mu\Big(\sum_{i=1}^{n_c-\widetilde{n}_c}\Lambda_i\psi_i + \frac{1}{2}\text{tr}\phi^2\Big),$$

$$(6.6)$$

for an $N=1$ $SU(\widetilde{n}_c)$ theory. In addition to some flat directions similar to those in (6.5) in which the quarks in the fundamental of the unbroken gauge group are all lifted and there are various massless mesons, in this case there are also vacua where the e_k get vevs, Higgsing all the $U(1)$ factors. Integrating out these massive fields, we find the effective

superpotential

$$\mathcal{W}'_{\text{bb}} = \sqrt{2}\text{tr}(q\phi\widetilde{q}) + \frac{\mu}{2}\text{tr}\phi^2, \tag{6.7}$$

describing $SU(\widetilde{n}_c)$ super–QCD with n_f flavors in the limit $\mu\rightarrow\infty$, thus showing the origin of the "magnetic" gluons and quarks of the dual $N{=}1$ theory of [7].

Acknowledgments

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